Bot Got Your Tongue?
Social Learning with Timidity and Noise

John W.E. Cremin*†

April 21, 2026

Latest Version

Abstract

Models of social learning conventionally assume that all actions are visible, whereas in reality, we can often choose whether or not to advertise our choices. In this paper, I study a model of sequential social learning in which *social* agents choose whether or not to let successors see their action, only wanting to do so if they are sufficiently confident in their choice (they are *timid*), and *noise* agents act randomly. I find that in *sparse* networks, this produces a form of unravelling to the effect that noise agents are overrepresented. This can damage learning to an arbitrary extent if social agents are sufficiently timid. In *dense* networks, however, no such unravelling occurs, and the combination of noise and timidity can facilitate complete learning even with *bounded* beliefs.

Keywords— Sequential Social Learning, Endogenous Social Networks, Network Theory, Information Economics

1 Introduction

To brag or not to brag? That is an often pertinent question. Having undertaken a particular purchase, made one's mind up on the political controversy du jour, or chosen to invest in a certain stock, we are often able to *choose* whether or not our peers or colleagues are aware of our choice. Models of social learning, however, conventionally neglect this fact and assume that agents make a decision, and will then be observed by their neighbours whether they like it or not. Whenever agents benefit or suffer upon being seen to act wisely or unwisely, perhaps through reputation effects or mockery from peers, and have some ability to vary the observability of their action, this assumption is not anodyne. In a world where the visibility of one's action is a choice, any observation network becomes endogenous, and the information a given action carries is altered by the fact that the agent in question *chose* to act visibly.

*Aix-Marseille School of Economics, Université d'Aix-Marseille, CNRS, john-walter-edward.cremin@univ-amu.fr.

†I would like to thank Evan Sadler for his guidance with this project, as well as Navin Kartik, Jacopo Perego, Sebastian Bervoets and Romain Ferrali for many helpful comments and suggestions. I would also like to thank seminar participants at Columbia and Aix-Marseille Universities for their feedback. Finally, thanks to the Organisers of the ASSET 2025 Conference for awarding this paper the Louis-André Gerard-Varet Prize.

In this paper, I model agents who only make their action visible to others when sufficiently confident that it is correct, in a classical sequential social learning model à la Acemoglu et al. (2011).[1] Having introduced this *timidity* into the model, I consider its interaction with the presence of non-strategic noise agents who could for example represent partisans in political debate, or bots on social media platforms. Their combined effects on first learning and second the overrepresentation of noise agents are then shown to vary as a function of whether the network is sparse or dense, where I define a sparse network as one in which we can find a uniform finite upper bound on the size of all agents' neighbourhoods.

Sparse networks turn out to be vulnerable to the presence of timidity and noise, for a number of reasons. Firstly, I establish that in these networks we will observe a form of *unravelling* in which timid and uncertain *social* agents (i.e. those agents that are not noise agents) will 'drop out' of the game by acting invisibly, and in so doing provoke yet more social agents to do the same, creating a vicious cycle. I characterise a lower bound to the severity of this effect in Theorem 2, and an upper bound on learning that follows from it in Proposition 5 and Corollary 2.1. After establishing some comparative statics to this first bound in Lemma 3, I can then demonstrate that introducing *enough* timidity can cause an arbitrarily high proportion of social agents to drop out, and smother social learning completely as in Proposition 2. Examples 1 and 2 display the effect of changing the timidity distribution in sparse networks, and help illustrate the breakdown of the *improvement principles* on which much of the literature depends to establish learning in such networks. They also highlight the trade-off between observing more noise agents and more confident social agents that timidity introduces.

Dense networks, conversely, do not exhibit any unravelling. Moreover, it is possible that timidity and noise together will actually help matters by removing the *cascade beliefs*, which are social beliefs at which an agent will choose the same action regardless of their private signal, and facilitating learning. Taking the complete network, I show in Theorem 1 that timidity and noise can deliver learning with bounded beliefs. In this case, we can always find timidity distributions (those with support containing both the very confident *and* the very timid) in which agents converge to certainty on the true state, though this does not occur in their absence. This is due to the fact that without noise agents, at a given cascade belief all agents will choose the corresponding action with probability one regardless of the state, but at least half of noise agents, when they are present, will choose the opposite action. The probability of observing each action will then depend on the likelihood of social agents dropping out, which will be less likely to occur if the cascade belief supports the true state than otherwise since private signals will be less likely to produce moderate beliefs. If the timidity distribution satisfies the condition mentioned above, action one will be strictly more likely if the state is one than if it is zero for any interior social belief. This guarantees learning, and removing either timidity or noise is thus enough to prevent it in this setting.

Literature Review: There is an extensive literature on social learning with a binary state, much of this considers only the complete network (Banerjee, 1992; Bikhchandani et al., 1992; Smith and Sørensen, 2000). Various articles develop this literature by exploring learning on more general network topologies, and my approach follows the workhorse model of Acemoglu et al. (2011), in which it resembles Lobel

[1] Bikhchandani et al. (1992), Banerjee (1992) and Smith and Sørensen (2000) are earlier (seminal in the case of the first two) references focusing on complete networks, I consider more general networks topologies as in Acemoglu et al. (2011).

and Sadler (2015, 2016); Cremin (2025); Lomys (2020). Lobel and Sadler (2016) and Cremin (2025) both consider models in which the classic proof technique of the *improvement principle* breaks down, due in the former to agents having different tastes, and in the latter their rejection of social information via motivated reasoning. The presence of noise agents here has a similar impact to that of motivated reasoners in Cremin (2025), though the fashion in which social information is lost is much more blunt with the former. In Lobel and Sadler (2016), instead the problem is that when unaware of the tastes of a given neighbour it is no longer possible to straightforwardly improve on their action. In copying an agent with different tastes, one does not achieve the same utility, and thus 'improving' on their action with a new private signal may nonetheless yield a lower utility.

A distinct literature extends the original complete network models by varying the visibility of earlier actions.[2] Herrera and Hörner (2013) and Guarino et al. (2011) both model learning when only one of the two actions is visible, therefore making the observation network endogenous. In the first case arrival time is random, and in the second agents do not know at what point in the sequence they arrive. My paper is quite different to these in that the visibility of actions is a choice. Beyond these Song (2016) studies a model in which agents are able to choose which predecessors they observe at some cost, and in which the network topology is also therefore endogenous, but agents cannot choose whether or not their action is observable by successors. To the best of my knowledge, no other papers yet allow agents to themselves choose the visibility of their action, and make it a function of their degree of confidence that they are correct. This seems an important feature of social learning in the real world, in many contexts, and thus forms part of the motivation of this paper.

The paper is organised as follows. In Section 2, I set out the model, and present some lemmas characterising the decision rules of agents in Section 2.1. I then move on to discussing my results in Section 3, which includes learning in dense and sparse networks, unravelling, and benchmark results. In Section 3.1, I show how timidity and noise can help learning in dense networks when agents have bounded beliefs with Theorem 1, which also shows that in the complete network they make no difference when signals are unbounded. Section 3.2 then demonstrates the unravelling effect that occurs in sparse networks, providing a lower bound on the fraction of visible agents who are of noise type in Theorem 2. Proposition 1 gives sufficient conditions for such an unravelling to occur (where the fraction of visible agents who are of noise type will almost surely be strictly greater than in the general population). I then discuss learning in sparse networks further in Section 3.3 with Examples 1 and 2. Section 4 considers to what extent my results depend on the precise model specification, or are robust to extensions.

[2]Models in which agents observe some summary statistics of their predecessors' actions have also been studied, e.g. Guarino and Jehiel (2013), Callander and Hörner (2009), but the connection between those and the model of this paper is much weaker.

2 The Model

We have a binary state $\theta \in \{0,1\}$, which nature draws uniformly at the beginning of the game. The common prior of all agents reflects this, and assigns 0.5 to each state.[3] An infinite sequence of agents arrive at exogenously given times, their indices representing their position in this sequence $n \in \mathbb{N}$, with agent 1 arriving after Nature draws the value of the binary state. Upon arrival, each agent observes some information about the true state of the world $\mathcal{I}_n$, and decides both which action to take $x_n \in \{0,1\}$, and whether or not to make this decision visible $v_n \in \{0,1\}$. Agents come in two unobservable types, *noise* and *social* $\tau_n \in \{N,S\}$. Noise types do not attempt to match their action to the state, but simply randomise uniformly between the two actions, and act visibly $v_n = 1$. Social types use their information to try and match their action to the state, and prefer to have acted visibly if their action is correct, but invisibly if not. The precise details of this can be seen in the utility functions I present momentarily, and each agent is a noise type independently with probability ρ.

Before choosing their action, each agent receives a conditionally independent private signal s_n (this could represent simply intuition, or the result of individual research). The private signal is described by the information structure $(\mathbb{F}_0, \mathbb{F}_1)$ (where $\mathbb{F}_\theta$ is the distribution of the private signal in state of the world θ). I assume these belief distributions do not contain a perfectly revealing signal (are mutually absolutely continuous with respect to each other), but are informative ($\frac{d\mathbb{F}_1}{d\mathbb{F}_0} \neq 1$ on a non-null set). These private signal distributions imply private belief distributions $p_n \sim \mathbb{G}_\theta$, that give the distribution of the posterior belief an agent would form upon observing only private information.

In addition to this, I assume each agent observes the actions of some subset of those agents who have visibly-acted before them (chosen $v_n = 1$). Each agent's order in the sequence is of course given by their index, but we can also define an *adjusted index* for each to reflect the number of visible actions that have been taken at the time that agent arrives: $\tilde{n} := 1 + \sum_{i=1}^{n} v_i$. Agents do not observe either their own index or those of their neighbours, but they do observe their adjusted index and those of their neighbourhood: each agent with adjusted index $\tilde{n}$ observes neighbourhood $\{x_{\tilde{k}} : \tilde{k} \in B(\tilde{n})\}$.[4] These neighbourhoods are drawn according to the commonly-known distributions $\{\mathbb{Q}_{\tilde{n}}\}_{\tilde{n}\in\mathbb{N}} = \mathbb{Q}$ at the beginning of the game, and are independent across $\tilde{n}$. Note that multiple agents will, in general, have the same adjusted index, though only one visibly-acting agent has each. The network is based upon these adjusted indices $\tilde{n}$, and not one's position in the overall sequence of arrivals, n. I shall define network topologies for which there is some $M \in \mathbb{N}$ such that $|B(\tilde{n})| < M$ almost surely as *sparse*, and networks in which this is not the case as *dense*.[5]

[3] This assumption is to simplify notation, the fundamental patterns explored in the paper do not require an even prior.

[4] For applications regarding social media (where the binary visibility choice represents the decision to comment or not) this assumption is particularly defensible, since it amounts to assuming agents can observe how many predecessors have commented on a given hashtag or topic already, without knowing the number of people who have actually logged on and considered a matter without commenting. In real world learning problems, assuming that agents do not even observe their adjusted index may be more appropriate. Fortunately, the unravelling results of this paper generalise beyond this exact specification to include this case.

[5] I am largely thinking as sparse networks as having some 'small' M upper bound on neighbourhood size, and as can be seen in the results, the unravelling effect I discuss becomes less and less severe the larger this M gets. In addition, I mainly consider deterministic networks in my examples, though define these terms probabilistically for full generality: the results all apply to this general case.

Upon observing this social information, agents form their social beliefs sb_n: in each state of the world, a profile of equilibrium strategies induces a probability distribution over histories, and thus the actions of agents in $B(\tilde{n})$, so forming these beliefs involves simply an application of Bayes' rule. One can partition the set of histories (infinite or after a finite amount of time) into sets such that the jth visible action is the same for each history in a given element of this partition for every j, and the number of visible actions is also the same. I shall call each element of this partition a *visible history* h^v, and it will be convenient at points to discuss agents acting after a given visible history, meaning after any history within that element of the partition. Hence I shall use $n(h^v)$ and $\tilde{n}(h^v)$ to refer respectively to the first agent and visible agent who act at history h^v in a given equilibrium (they may be the same agent if the first to arrive chooses to act visibly).[6] Upon arriving in the game and observing their adjusted index, I assume agents do not update their beliefs about θ. I discuss this assumption and its relation to the *Sleeping Beauty problem* further in Section 4, and argue that it is correct for Bayesian agents in this setting.

Given their posterior belief (formed with both private and social information) a social agent n chooses (x_n, v_n) to maximise:

$$u_n(x_n, v_n, \theta) = \begin{cases} 1 + v_n & \text{if } x_n = \theta, \\ -\left(1 + \frac{1}{c_n} \cdot v_n\right) & \text{if } x_n \neq \theta, \end{cases} \tag{2.1}$$

$c_n \in (0, 1]$ is a parameter representing the agent's 'confidence', drawn i.i.d. for each agent from some distribution $\Delta(c)$. Social agents are always trying to match the true state of the world, but their payoffs are more extreme for visible actions, though asymmetrically so. They dislike more to be caught publicly saying something incorrect than vice versa, and the extent of this asymmetry is decreasing in their confidence. I further define a social agent's 'timidity' as the inverse of their confidence $t_n := \frac{1}{c_n} \sim \Delta(t)$. If $c_n = 1$ a social agent will never choose an invisible action $v_n = 0$, but if we let $c_n \to 0$ they will certainly do so. A social agent choosing $x_n = 1$ will choose for this action to be visible if their posterior is that the true state is 1 with probability strictly greater than $\frac{1}{1+c}$.[7] Figure 1 represents this. The exact form of this utility function is not essential; there are two properties for which it has been chosen: (1) agents would like to match their action to the true state, come what may; (2) agents only want to act visibly if they have strong beliefs. Even then, this second property is not essential. The unravelling argument of section 3.2 requires only that there is a non-zero probability that any agent observing a unanimous neighbourhood (M neighbours all choosing $x = 1$, or $x = 0$) forms an overall belief that does not provoke a visible action. Hence so long as some open interval around 0.5 induces $x = 0$, the unravelling reasoning is unaffected. For Theorem 1, we need only that the probability of an agent choosing $x = \theta$ visibly is strictly higher in state θ than $\neg\theta$; any utility function that achieves this will suffice.

The solution concept employed in this paper shall be that of Perfect Bayesian Equilibrium, and *complete learning* shall be said to obtain if the probability with which agents match the state (their *accuracy* α_n) converges to 1 in every equilibrium. All omitted proofs can be found in the appendix.

[6] The identity of the agent that acts at a given visible history depends upon the equilibrium, so it is with some abuse of notation that I suppress this dependence here. None of the results of this paper require such notation, so I omit it.

[7] I break indifference here in favour of invisibility for notational convenience, it does not matter. I shall also

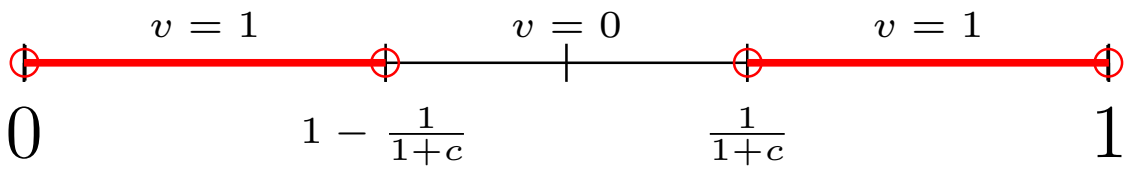


Figure 1: A Social Agent's Action Choices: The position on the number line represents the posterior belief of the agent. If their belief is greater than $\frac{1}{2}$ (that the true state is 1) they will choose $x_n = 1$, and $x_n = 0$ otherwise. In the red regions, their belief is confident enough to choose a visible action, otherwise they will choose an invisible action.

2.1 Decision Rules

A social agent's decision rule for choosing x_n can be represented in terms of the sum of the private and social beliefs, exactly as in Acemoglu et al. (2011, Proposition 2):

Lemma 1. *After any history, h_n, agent n will choose $x_n = 1$ if*

$$\mathbb{P}(\theta = 1|s_n) + \mathbb{P}(\theta = 1|B(n)) > 1$$

and $x_n = 0$ if this sum is strictly less than 1. If they exactly equal 1, the agent is indifferent and assumed to randomise uniformly.

This lemma is derived only assuming that the private and social signals are conditionally independent of each other, and does not depend at all on the network topology or the assumptions Acemoglu et al. (2011) make about it in the rest of their paper. We can similarly derive a rule governing the agent's choice of action visibility, which of course reflects that stronger beliefs are necessary to choose visible actions:

Lemma 2. *After any history h_n, the agent chooses $v_n = 0$ if:*

$$1 - (t-1)\mathbb{P}(\theta = 1|B(n))\mathbb{P}(\theta = 1|s_n) < S_n < 1 + \frac{t-1}{t}\mathbb{P}(\theta = 1|B(n))\mathbb{P}(\theta = 1|s_n)$$

for $S_n = \mathbb{P}(\theta = 1|B(n)) + \mathbb{P}(\theta = 1|s_n)$.

We can clearly see that the threshold for choosing $v_n = 1$ when $S_n > 1$ is higher, since added to the original threshold is a positive cross-product of the social and private beliefs of the agent.

3 Results

As shall become clear, the phenomena of interest in this model result from the interaction between the timidity of social agents and the presence of noise agents. Whereas they can together be expected to limit

assume they randomise uniformly between $x_n \in \{0, 1\}$ if they form belief 0.5 similarly.

learning and give noise types an exaggerated presence amongst visibly-acting agents in sparse network topologies, they conversely work together to make it possible in very dense networks. With respect to learning, this contrast results from the fact that whereas in sparse networks agents depend on a few neighbours to learn about the history of visible actions up to that point, in very dense networks this is less the case (and not at all the case in the complete network) as they will be able to observe much of this history themselves. The combination of timidity and noise agents interferes with the improvement-based learning that achieves complete learning in the first case, but can remove the interior cascade beliefs that prevent it in the second.[8] When we consider the overrepresentation of noise agents, I shall analyse an unravelling effect that occurs in sparse networks (where there is a commonly-known bound M on each agent's number of neighbours), and becomes more extreme the more sparse the network is (the lower this M), but disappears if there is no commonly known integer bound on neighbour numbers ($M = \infty$). These results together show how the impact of noise and timidity depends on the density of the network: in *dense* networks they can help by removing cascade beliefs, in *sparse* networks they produce unravelling, which undermines learning and produces an overrepresentation of noise.

3.1 Learning in Dense Networks

In dense networks, first of all, the situation is quite different to that in the sparse setting. Firstly, if there is no finite common bound on the number of neighbours each agent has, $M = \infty$, the unravelling we will discuss in Section 3.2 does not occur. Moreover, agents are generally not so dependent on recent neighbours to learn about the history of the game, since they can directly observe much of it. Conversely, belief cascades become more concerning, and produce incomplete learning in the canonical model (Smith and Sørensen, 2000) when we have bounded signal structures. As I show in the following, using the complete network as a tractable example of a dense network topology, the combined presence of timidity and noise agents can actually help learning by reducing (or completely removing) the set of cascade beliefs (beliefs at which agents simply ignore their private signals (Smith and Sørensen, 2000)).

[8] *Improvement principles* prove learning by noting that in equilibrium, a Bayesian agent must do at least as well as an agent who simply copies their most accurate predecessor, only disagreeing upon receipt of a sufficiently extreme opposing private signal. Cascade beliefs are social beliefs strong enough that agents will act according to them whatever private signal they receive; they are only possible with bounded private signals.

Theorem 1 (Learning in the Complete Network)**.** *The following two statements characterise learning in the complete network:*

1. *With unbounded beliefs, there is complete learning. The social belief converges to certainty on the true state almost surely.*

2. *With bounded beliefs supported on* $[\underline{p}, \overline{p}] \subset (0,1)$*, and a confidence distribution* $\Delta(c)$ *such that for some* $\epsilon > 0$*,* $(0, c^* + \epsilon) \cup (1-\epsilon, 1) \subseteq \text{supp}(\Delta(c))$*, where*

$$c^* := \frac{\underline{p}(1-\overline{p})}{\overline{p}(1-\underline{p})},$$

the social belief converges to certainty on the true state almost surely. Otherwise complete learning almost surely does not obtain.

The first part of this theorem simply establishes that in the complete network with unbounded beliefs the presence of timidity and noise agents makes no difference. The second part, however, gives the aforementioned setting in which timidity and noise help: for any bounded belief setting we can find some confidence distribution that produces complete learning. Precisely, timidity helps learning here by providing a signal of the confidence with which agents believe the state of the world matches their action. When the distribution of this confidence parameter has support sufficiently close to 0, it is no longer the case that neighbour actions can be completely uninformative for social beliefs beyond a certain point. One could write a simpler, though weaker, theorem statement by simply assuming the confidence distribution is full support; in the following I explain why the chosen assumption is the minimal necessary one. The threshold c^* is determined by the endpoints of the private belief support; requiring confidence values below $c^* + \epsilon$ to be supported ensures that for every social belief, some agents are timid enough that their visibility thresholds (c.f. Lemma 2) fall strictly inside $(\underline{p}, \overline{p})$. This guarantees that there is either strictly positive probability that a social type varies either x_n, v_n, or both upon observing different private signals.[9] However, it is not just *timidity* helping here, as if we removed the noise agents but left timidity we would once again have incorrect learning. In a model in which agents could observe that an agent had decided to act invisibly, timidity would suffice; this is because the problem with cascade beliefs is that an agent with such a belief will always choose the same course of action irrespective of their private signal. This ensures that this action reveals no information about said private signal, and so information ceases to accumulate. If we had no noise agents but could observe when an agent had just chosen to act invisibly, this choice to act invisibly (for a social belief that is strong enough in favour of $\theta = 1$ that choosing $x = 0$ is a probability 0 event) would serve as a noisy signal of that agent's private belief. One way of seeing this is to observe that some social beliefs would be cascade beliefs vis-à-vis the action itself, but not cascade beliefs in terms of an agent's choice of visibility: a social belief would need to be a cascade belief in both

[9] This almost sure convergence to the truth with bounded beliefs and sufficient timidity is reminiscent of Goeree et al. (2006, Theorem 2), where my assumption that the support of the confidence parameter contains $(0, c^* + \epsilon)$ for some $\epsilon > 0$ performs the same function as their assumption 3, that the joint distribution of private values over the A available actions has full support: $\text{supp}(f^t) \supseteq [0,1]^A$. Both ensure that the martingale social belief process can only settle on degenerate beliefs.

senses to stop learning. Of course, in this model agents cannot observe when others have chosen to act invisibly. If there were no noise agents, at cascade beliefs we would simply observe an infinite sequence of all visible agents taking the action consistent with that social belief. No information would be revealed about the proportion of agents choosing to act invisibly. Since there are noise agents, however, there is a minimum probability of $\rho/2$ that each action will be chosen, since it is with this probability that noise agents arrive and choose each action. The proof proceeds by showing that the probability $\phi(sb, \theta)$ with which the next visible agent chooses action 1 is strictly higher in $\theta = 1$ than $\theta = 0$ for every interior social belief. This is established by decomposing ϕ into contributions from agents who are visible and choose 1 (at rate R_θ) and agents who are visible and choose 0 (at rate L_θ), and showing that the first order stochastic dominance of private beliefs in state 1 over state 0 ensures $R_1 > R_0$ or $L_1 < L_0$ (or both) for every interior social belief. The condition on the confidence distribution ensures that for every social belief, a positive measure of social agents (or of their timidity values speaking precisely) have visibility thresholds inside the private belief support, where this dominance is strict. As the social belief converges to certainty on the true state, the probability with which the next visible agent is a noise agent will converge to ρ, as all social agents become confident enough to act visibly.

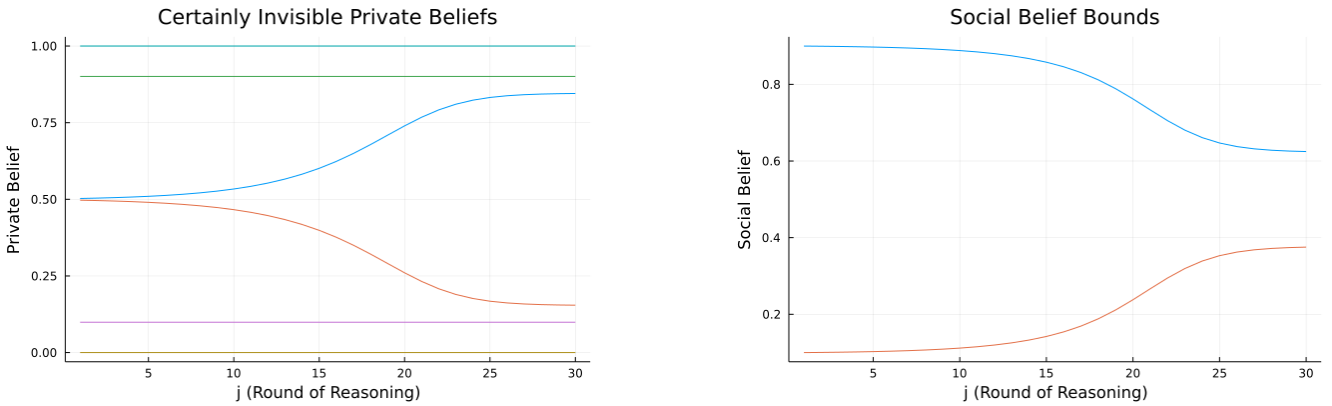
In addition to allowing complete learning with the assumptions of Theorem 1, the timidity distribution can also serve to simply reduce the size of the set of cascade beliefs. Whereas in sparse networks, as we shall see shortly, learning is generally damaged by timidity and noise, in dense networks they can produce learning for any non-degenerate private signal distribution.

3.2 Unravelling

Even in the absence of timidity, the bound on neighbourhood size itself limits the maximum possible strength of social beliefs (to see this formally, consider Proposition 5 in Section 3.4). We shall now see that this can produce much tighter constraints on learning and a much greater presence for noise agents than it may at first seem. An implication of these bounds on social beliefs is that if agents are sufficiently timid, there will be private signals for which they certainly (whatever their social signals) choose to act invisibly. Specifically, using Lemma 2 we can see that private signals within the following interval certainly produce invisible actions, if t_n is large enough for it to be nonempty:

$$\mathbb{P}(\theta = 1|s_n) \in [\underbrace{1 - \frac{t_n(\frac{\rho}{2})^M}{t_n(\frac{\rho}{2})^M + (1-\frac{\rho}{2})^M}}_{L_0}, \underbrace{\frac{t_n(\frac{\rho}{2})^M}{t_n(\frac{\rho}{2})^M + (1-\frac{\rho}{2})^M}}_{U_0}]$$

Suppose at first that $t_n = t$ for all n. If we denote $\lambda_0 := \min\{\mathbb{G}_0(U_0) - \mathbb{G}_0(L_0), \mathbb{G}_1(U_0) - \mathbb{G}_1(L_0)\}$, we can see that whatever the state of the world, different agents will form private beliefs within this region with at least probability λ_0, where of course agents' private beliefs are conditionally independent. Since neighbours are drawn only from the $v_n = 1$ pool, if one's neighbourhood is drawn from agents with at most social beliefs in the region $[\underline{SB}, \overline{SB}]$ the type distribution of one's neighbourhood is no longer $\rho, 1-\rho$, since at most $(1 - \lambda_0)$ of the social agents choose visible actions. Instead, the relevant distribution is one with lower bound noise-probability $\underline{\rho}_1 := \rho/\big(\rho + (1 - \lambda_0)(1 - \rho)\big)$.



(a) (b)

Figure 2: In panel (a), the curved blue and orange lines give the bounds of the interval of private beliefs that must produce invisible actions. This interval clearly expands with each round of reasoning. The green and purple lines bounding them give the private signals that are sufficient for a visible action even with a social belief of 0.5. In panel (b), the corresponding interval of possible social beliefs collapses. The parameters in this example are $f_1(x) = 2x$, $f_0(x) = 2 - 2x$, $M = 1$, $\rho = 0.2$, $t \approx 9$, $\underline{\rho}_\infty = 0.754 (3d.p.)$.

This reasoning, however, can be iterated. The upper and lower bounds on the social beliefs of an agent with such a neighbourhood are tighter since the neighbours they observe are more likely to be noise agents. Thus if an agent's neighbourhood is drawn from a set of agents whose social beliefs are necessarily within $[\underline{SB}_1, \overline{SB}_1]$, we can define a new private belief interval $[L_1, U_1]$, a new λ_1, and thus a $\underline{\rho}_2$, all by substituting $\underline{\rho}_1$ in place of ρ. Iterating this reasoning infinitely many times, at each stage we get an at least weakly tighter range of social beliefs, and a weakly higher $\underline{\rho}_j$.[10] Figure 2 shows an example; the region of private beliefs that must produce an invisible action can be seen to expand, and correspondingly the range of possible social beliefs shrinks. Taking $\underline{\rho}_j$, this is clearly a weakly increasing and bounded sequence (since ρ is a probability) and thus converges. Fundamentally, what this line of reasoning provides is a fixed-point interval of social beliefs. If you observe neighbours with social beliefs in $[\underline{SB}_k, \overline{SB}_k]$, you must have a social belief within $[\underline{SB}_{k+1}, \overline{SB}_{k+1}]$, and thus $[\underline{SB}_\infty, \overline{SB}_\infty]$ is that social belief interval (which notably contains the starting social belief $\frac{1}{2}$) that cannot be 'escaped' by a society. Thus the presence of timidity can produce a tighter range of possible social beliefs.

Without imposing that all agents have a single t, we can apply similar reasoning for any distribution $\Delta(t)$, defining the functions $L_j(t)$ and $U_j(t)$, that give the certainly-invisible private belief regions for each t. Letting t_j^* be the value of t such that $L_j(t) = U_j(t)$, we can similarly define λ_j as the analogous lower-bound probability of an invisible action, precisely with the integral:

$$\lambda_j = \min \left\{ \int_{t_j^*}^{\infty} \mathbb{G}_0(U_j(t)) - \mathbb{G}_0(L_j(t)) d\Delta(t), \int_{t_j^*}^{\infty} \mathbb{G}_1(U_j(t)) - \mathbb{G}_1(L_j(t)) d\Delta(t) \right\}$$

Once again this produces a weakly increasing and bounded sequence $\{\underline{\rho}_j\}_{j \in \mathbb{N}}$ that converges to a bound

[10] Indexing the rounds of this reasoning by j

$\underline{\rho}_\infty$. The iteration process with a distribution of timidity levels is illustrated by the example in Figure 3; the black lines give $L_0(t)$ and $U_0(t)$, and thus contain the (p_n, t_n) pairs that certainly engender invisible actions for agents with a social belief in $[\underline{SB}_0, \overline{SB}_0]$. The purple line gives the 100th round of this reasoning, and the red the 150th round, illustrating the fact that this region grows and converges to a larger region than the original. The brown line shows the probability density function of the assumed timidity distribution.

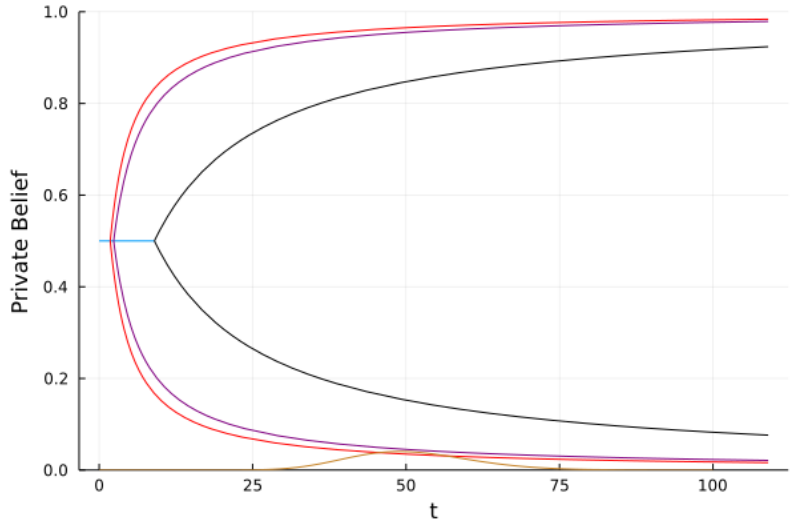


Figure 3: The parameters for this example are $f_1(x) = 2x$, $f_0(x) = 2 - 2x$, $M = 1$, $\rho = 0.2$; the distribution of $t_n - 1$ is a Chi-squared with 50 degrees of freedom.

Theorem 2. $\underline{\rho}_\infty = \frac{\rho}{\rho + (1-\lambda_\infty)(1-\rho)}$ *is a lower bound on the probability that the* $\tilde{n}$*th visibly-acting agent is a noise type for any* $\tilde{n} \in \mathbb{N}$*, and the asymptotic fraction of agents that are noise types is almost surely greater than* $\underline{\rho}_\infty$.

Proof. The reasoning in the preceding paragraphs clearly establishes that for any set of agents whose social beliefs are within $[\underline{SB}_\infty, \overline{SB}_\infty]$, the probability that the first to visibly act is a noise agent is at least $\underline{\rho}_\infty$. The Strong Law of Large Numbers then gives that: (1) the asymptotic fraction of social agents choosing invisible actions is almost surely at least λ_∞, since private signals are independent and identically distributed, and (2) the asymptotic fraction of agents who are of noise type is almost surely ρ, since types are also independent and identically distributed. Thus the asymptotic fraction of agents who are social and act visibly is at most $(1 - \lambda_\infty)(1 - \rho)$, and the asymptotic fraction of agents who are noise agents is ρ. The remainder are social agents who act invisibly, thus the fraction of visibly-acting agents who are noise types is at least $\rho / \big((1 - \lambda_\infty)(1 - \rho) + \rho\big) = \underline{\rho}_\infty$. □

Note also that this reasoning applies if agents observe their own adjusted index and those of the agents they observe (as mimics Acemoglu et al. (2011)), only their own adjusted index (Smith and Sørensen, 2008), or neither (Monzón and Rapp, 2014). Hence this unravelling will occur in a larger class of social learning problems than that explicitly studied in this paper. Additionally, although the inequalities discussed thus far have been predominantly weak, we do not need strong conditions to produce strict ones (and thus an actual unravelling effect):

Proposition 1 ($\underline{\rho}_\infty$ Properties)**.** *$\underline{\rho}_\infty > \rho$ if and only if $\lambda_0 > 0$, which is guaranteed by assuming that $\Delta(t)$ assigns mass $\epsilon > 0$ to t values strictly greater than t_0^*, and that the interval $(0.5-\delta, 0.5+\delta)$ is contained within the support of private beliefs for some $\delta > 0$.*

We can arrive at the stronger statement that $\underline{\rho}_j > \underline{\rho}_{j-1}$ for any $j \in \mathbb{N}$ by assuming that the t distribution, $\Delta(t)$, is full support; that the private beliefs distribution is full support, and some positive mass $\epsilon > 0$ is assigned to timidity values greater than t_0^*; or finally that private beliefs are unbounded, and the timidity distribution assigns some mass $\epsilon > 0$ to $t_n > \min\{L_0^{-1}(\underline{p}), U_0^{-1}(\overline{p})\}$ where $\overline{p}, \underline{p}$ are respectively the highest and lowest private beliefs with density 0.

Beyond noting characteristics of this fixed point, one can observe the immediate implication that it provides a tighter bound on accuracy than we had already established in the absence of timidity:

Corollary 2.1 (Corollary to Proposition 5 and Theorem 2)**.** *For a commonly known bound M on the maximal number of neighbours, and the $\underline{\rho}_\infty$ implied by this M and Theorem 2, the accuracy of all agents is bounded above by:*

$$\alpha_n \leq \frac{1}{2}\mathbb{G}_0\left(\frac{(1-\frac{\underline{\rho}_\infty}{2})^M}{(1-\frac{\underline{\rho}_\infty}{2})^M + (\frac{\underline{\rho}_\infty}{2})^M}\right) + \frac{1}{2}\left(1 - \mathbb{G}_1\left(1 - \frac{(1-\frac{\underline{\rho}_\infty}{2})^M}{(1-\frac{\underline{\rho}_\infty}{2})^M + (\frac{\underline{\rho}_\infty}{2})^M}\right)\right)$$

It does not necessarily follow from this that actual asymptotic accuracy will be lower with timidity than without it, and I shall discuss this further in the next couple of propositions (and more in Section 3.3), which I establish using the comparative statics laid out in this lemma:

Lemma 3 ($\underline{\rho}_\infty$ Comparative Statics)**.** *$\underline{\rho}_\infty$ is increasing in ρ, increasing with First order stochastic shifts in $\Delta(t)$, decreasing in M, and decreasing with any mean-preserving spread in $\mathbb{G}_0$ or $\mathbb{G}_1$.*

Now of course, since these are the comparative statics of a bound, they do not necessarily reflect the comparative statics of the actual asymptotic fractions of agents who are noise agents, much as Corollary 2.1 does not necessarily reflect that asymptotic accuracy decreases as we introduce timidity.[11] The use of this lemma is rather in deriving the next two propositions. Proposition 2 states that in sparse networks, increasing timidity will always eventually harm learning, and achieve complete domination by noise agents. The fourth of these points, concerning the spread of the private belief distributions, in turn leads us on to Proposition 3.

Proposition 2 (Unbounded Analogue to Proposition 4)**.** *Suppose we have a learning game with $M < \infty$, and consider a sequence of timidity distributions $\Delta^k(t)$ for $k \in \mathbb{N}$, in which each successive distribution is shifted to the right by some fixed value $\delta > 0$.*[a] *$\underline{\rho}_\infty^k \to 1$, and from this the limiting visible accuracy $\tilde{\alpha}^* \to 0.5$ and accuracy $\alpha^* \to \frac{1}{2}\mathbb{G}_0(0.5) + \frac{1}{2}\big(1 - \mathbb{G}_1(0.5)\big)$.*

[a] We could also write this result in terms of a sequence of FOSD shifts, so long as we insist that each term has at least $\delta_1 > 0$ mass shifted at least $\delta_2 > 0$ distance to the right (simply shifting the distribution to the right is of course an example of an FOSD shift).

[11] As can be seen in the next section, the fraction of agents who are noise type does not even necessarily converge. I present examples in which it does not.

This formalises the fact that whilst some timidity can actually help learning, increasing timidity too much is always harmful. Figure 5a in Section 3.3 provides an example that captures this intuition. A series of FOSD increases in the (shifted) Poisson timidity distribution successively reduces the level of asymptotic visible accuracy, though the first increase instead helps learning for a range of values of ρ. It is also worth noting, as I do in the proof of Proposition 2, that this convergence is faster the lower we set M. Hence it is not only true that sparse networks are vulnerable to timidity in the limit, but that the *more* sparse they are, the *sooner* increasing timidity will do this damage.

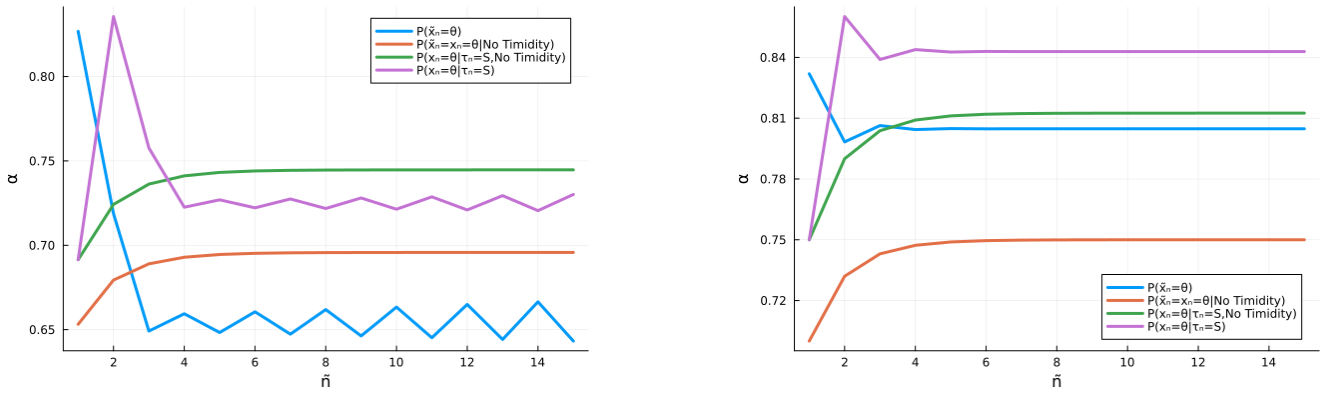
Turning our attention from shifting the timidity distribution to reducing the spread of the private belief distributions we have Proposition 3. Since we have a single timidity distribution in this result, we can define $\underline{t}$ as the lowest t value supported by this timidity distribution. Finally, for a sequence of private belief distributions $\{\mathbb{G}_0^k, \mathbb{G}_1^k\}_{k\in\mathbb{N}}$, define $U_\infty^k(\underline{t})$ and $L_\infty^k(\underline{t})$ as the corresponding series of functions evaluated at $\underline{t}$.

Proposition 3. *Consider a learning game with some $M < \infty$ and some set of private belief distributions $\{\mathbb{G}_0, \mathbb{G}_1\}$ such that $U_\infty^1(\underline{t}) > L_\infty^1(\underline{t})$. If we take a sequence of new unbounded belief distributions $\{\mathbb{G}_0^k, \mathbb{G}_1^k\}_{k\in\mathbb{N}}$ such that each is a mean-preserving spread of its successor, and the mass assigned to $[L_\infty^1(\underline{t}), U_\infty^1(\underline{t})]$ converges to 1, then $\underline{\rho}_\infty^k \to 1$.*

As with Proposition 2, this convergence will happen faster when M is lower. The fact that this last proposition places a constraint on the lowest supported t value does of course make it less widely applicable than Proposition 2, but it should be noted that similar convergence will occur without it, except that $\underline{\rho}_\infty^k$ will not converge to 1, but to some limit strictly below 1. So long as $[L_\infty(t), U_\infty(t)]$ is non-empty for some t and the mass of the $\mathbb{G}$ distributions converges to within the interval defined by some supported t, thinner tails will ensure that this lower bound $\underline{\rho}_\infty^k$ climbs upward.

3.3 Learning in Sparse Networks

As I have just established in the last section, when agents are learning in the presence of both noise types and timidity together, with *enough* timidity we can stamp out social learning to an arbitrary extent (though never quite preventing it completely if there are unbounded beliefs, as with bounded beliefs in Proposition 4). More generally, timidity can have two countervailing effects in the presence of noise agents. On the one hand, if social agents choose not to act visibly out of timidity and the proportion of noise agents observed increases, a smaller fraction of each agent's neighbours are informative in expectation; they observe more noise. On the other hand, those agents who are most likely to act invisibly are those with the weakest beliefs, i.e. those who are closest to being noise anyway. If the first agent to act at a given visible history is social, and chooses to act invisibly, the next could be either a noise type who acts visibly without information, or another social agent who receives a more informative signal and whose action provides a better indication of the state. Proposition 4 shows how with enough timidity this first effect completely dominates (and Proposition 3 shows that with thin enough tails on our private belief distributions and a timidity distribution that does not support low t values, this is also the case), but in general one can find examples where either the first or second effect dominates asymptotically i.e. one



(a) Normal Signals

(b) Signals with densities $\{2(1-s), 2s\}$

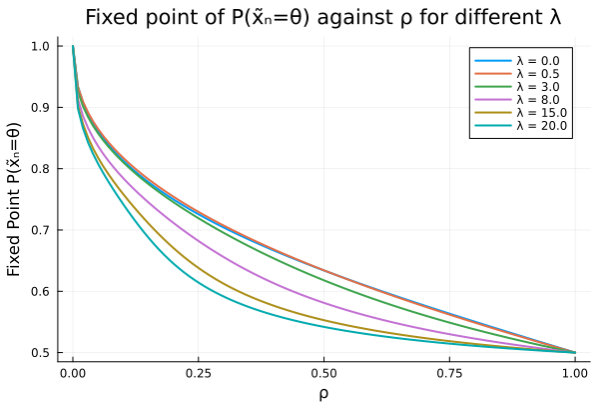
Figure 4: Graphs for Example 1

can find parameters such that agents match the state with a higher or lower probability in the limit when they are timid than if we modelled them all as having $t = c = 1$, as in the standard model (Acemoglu et al., 2011). Examples 1 and 2 illustrate this fact, with Figure 5a showing a case in which the second effect dominates when introducing a little timidity, before being quickly overwhelmed.

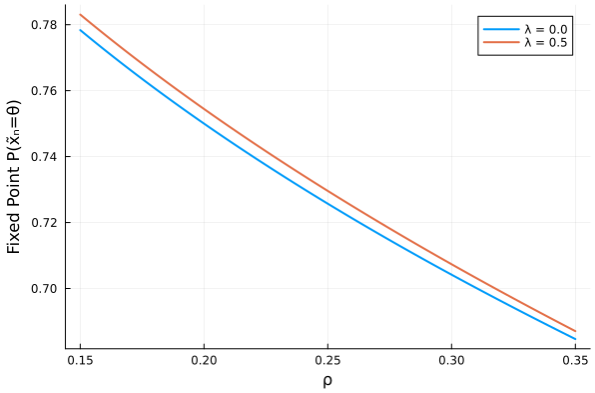
Example 1. *In this example, simply varying the private signal distribution and leaving all other parameters unchanged is enough to move from a setting in which (some) timidity harms asymptotic accuracy to one in which it helps. In both instances, assume that agents form an immediate predecessor network (each agent who arrives in the game sees only the agent who most recently acted visibly), have timidity that is either 1 or 82 with 0.5 probability each, and* $\rho = 0.2$. *In the first case, plotted in Figure 4a, the private signal distribution is normal, with variance* 1 *and mean* $\mu = \theta$. *In the second, plotted in Figure 4b the probability density functions are* $\{2(1-s), 2s\}$. *As is visible in the graphs, in the first case agents achieve higher asymptotic accuracy without timidity (it does not converge with timidity, but the limit without timidity exceeds the limsup with, so we can clearly say that they perform better). In the second however, removing timidity reduces their asymptotic accuracy, even though nothing has changed but the signal distributions, and in each case beliefs are unbounded. Whether or not timidity helps or hinders depends on the precise parameters assumed.*

Example 2. *In this example we have signals distributed according to densities* $\{2(1-s), 2s\}$, *but* $t - 1$ *follows a Poisson distribution with mean* λ. *The network topology is an immediate predecessor network, and I plot in Figure 5a the fixed point probability such that each visibly-acting agent performs exactly as well as his immediate predecessor. This shows that generally increasing the amount of timidity with a first-order stochastic shift in the timidity distribution reduces asymptotic accuracy for any value of* ρ, *and that accuracy in turn falls for any timidity distribution as we increase the proportion of noise agents. There is, however, a slight exception to the first of these points in that moving from* $\lambda = 0$ *(where all agents have* $t = 1$, *and thus are not timid) to* $\lambda = 0.5$ *increases accuracy for a large range of values of* ρ *(of which a portion is displayed in Figure 5b).*

It would be nice to establish general solutions that characterise the level of asymptotic accuracy



(a)

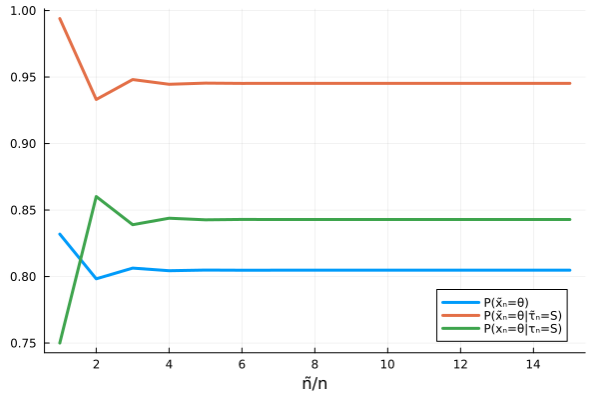


(b)

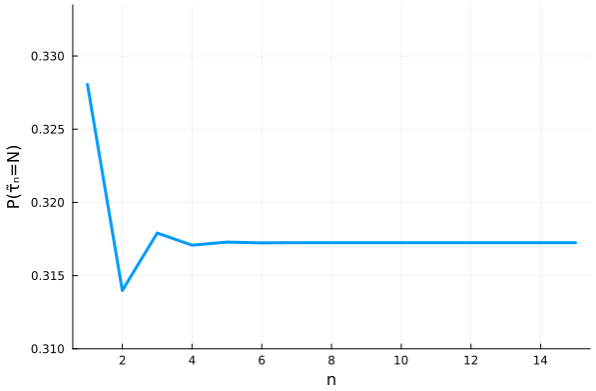
Figure 5: Graphs for Example 2

that should result from general classes of parameter assumptions here, but unfortunately the presence of noise agents, timidity and bounded neighbourhoods produces a number of analytical challenges that foil the standard tools of this literature. First of all, since we are interested here in sparse networks, martingale (for nested network topologies) and large sample techniques are of no use. Conventionally, in more general network topologies, one then resorts to improvement principles. These often allow the analyst to establish complete learning in sufficiently connected networks (those that satisfy Expanding Observations, see Definition 1 in the appendix), as is still the case when $\rho = 0$ (c.f. Theorem 3). However, the presence of noise agents and timidity together impede this technique here.

Figure 6a illustrates the problem, and a number of points can be gleaned from it. First of all, notice that improvement reasoning does enable us to observe that agents acting after a given visible history h^v (whether or not they act visibly) must match the state with higher probability than do the visible agents in their neighbourhood (c.f. Lemma 8 in Appendix B). Figure 6a shows an immediate predecessor network; the fact that the green line (the accuracy of social agents at that history, visible or not) is always strictly above the value of the blue line one step before (this shows the accuracy of visibly-acting agents) follows from this fact. We can also observe that the gradient of the green line is always the same sign as the gradient of said blue counterpart one step before; this follows from the fact that if one visibly-acting agent matches the state with a strictly higher probability than the one before, his successors will observe a signal (social and private combined) that Blackwell dominates the one he and other agents with his adjusted index received. Hence, they will perform better and the green line will climb. Having observed that the behaviour of the green line is very intuitive, however, we can note that the orange line (which tracks the accuracy of visible agents conditional upon their being social) is not at all so. The fact that it increases when moving between points 2 and 3 shows that when moving from a visible history at which agents are observing one signal, to another in which they are observing a *strictly Blackwell worse* signal, the probability with which they match the state actually goes up. At first glance this may seem to contradict Blackwell's theorem (alarmingly!), but it does of course not in fact do so. Blackwell's theorem (Blackwell, 1953) asserts that when one signal Blackwell dominates another, this is equivalent to saying that an agent observing the former is better off in their expected utility faced with any decision problem.



(a)



(b) The probability with which visible agent $\tilde{n}$ is a noise agent.

Figure 6

Here, however, we are not discussing the welfare of a single agent, but the probability with which the first agent who happens to act visibly matches the state (conditional on their being social in this case). Selecting any specific agent, we can assert from Blackwell's theorem that they will obtain higher expected utility and match the state with higher probability. Additionally, since we are conditioning on the event that the first agent to act visibly is of social type, we are making a statement about the tails of a belief distribution, rather than the welfare that results from acting on the basis of it in expectation. Figure 7 illustrates that providing a Blackwell-better signal does not necessarily increase the probability with which the next agent matches the state, even conditional on their being of social type.

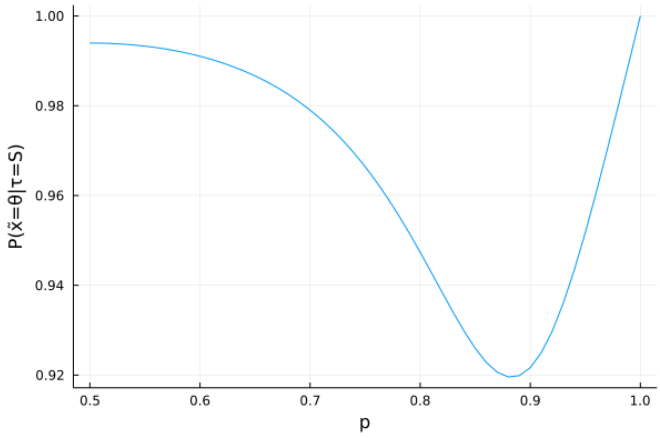


Figure 7: If at visible history h^v agents observe a Bernoulli trial with success probability p in place of a social signal, conditional on the first visible-acting agent being Bayesian, they match the state with probability given by this graph.

In addition to this, a more informative social signal can increase the probability with which the next visible agent is a noise type. An example of this is shown by Figure 6b, as the probability with which the second visible agent is a noise type is lower than the analogous probability that the first visible agent is, even though they evidently observe a Blackwell dominant social signal (the first agent observes none at

all). These two channels can both therefore lead to a given visible agent matching the state with lower probability than their neighbours. Improvement principles are thus no longer effective in this model, which reflects why timidity and noise are together harmful in sparse networks, where improvement principles reflect more closely the nature of the actual reasoning by which Bayesian agents choose their action.[12]

3.4 Timidity and Noise Benchmarks

Finally, it is worthwhile to consider the benchmark models in which we have timidity but no noise agents, and noise agents but no timidity. Taking the first of these, we can establish that the conditions that suffice for complete learning in the standard Bayesian model of Acemoglu et al. (2011) (that the network satisfies expanding observations[13] and the signal structure is unbounded) actually continue to guarantee it here:

Theorem 3. *With $\rho = 0$, unbounded signals and expanding observations are sufficient for complete learning in every equilibrium.*

Recalling the logic of the improvement principle behind this result: when expanding observations is satisfied, each agent will be at the end of an arbitrarily long *improvement path* in the limit, and at each step in any such improvement path the agent must be able to improve upon his predecessor.[14] Acemoglu et al. (2011) then prove that, with unbounded signals, the size of these improvements must be large enough that accuracy converges to 1 along these paths. Here, the presence of timidity simply ensures that each agent must asymptotically be at the end of an arbitrarily long improvement path of visible agents, where the improvement between each successive visible agent is greater than in the model without timidity. To clarify, if in the standard model (without timidity or noise agents) an agent observing a predecessor with accuracy α_b would have an accuracy at least Δ greater than α_b, in the model with timidity (but still no noise agents) we can find a lower bound on the improvement strictly greater than Δ. After any visible history h_n^v, it follows simply from the nature of Bayesian belief formation that the probability with which the next agent (visible or not) matches the state must be lower than the probability that the next agent with a strong enough belief to act visibly will do so; formally this last intuition is proved in Lemma 7.

Naturally this result does not suffice to establish that timidity is completely anodyne. Though expanding observations is a relatively minimal connectivity condition, without which complete learning is impossible with any signal structure and timidity distribution, the literature on social learning does not exclusively consider learning with unbounded signal structures. In the bounded setting, it is always possible to find a timidity distribution that completely prevents learning by assuming that all agents are at least timid enough to never act visibly with a social belief of 0.5. In this case, no social agent will ever be the first social type to act, and so knowing this all agents will forever have the social belief 0.5. This reasoning holds with or without noise agents; in their absence no agent ever visibly acts, in their presence exclusively noise agents do.

[12] In line networks, observing a predecessor and copying them unless one observes a sufficiently strong opposing private signal provides an exact description of what agents do. The more agents one observes, the worse an approximation this becomes.

[13] Definition 1 in Appendix A.

[14] An improvement path is a chain of agents in which each observes his predecessor in this chain.

Proposition 4. *For any bounded private signals, there is some* $\overline{c}$ *such that if* $\text{supp}(\Delta(c)) \subseteq [0, \overline{c}]$*, every agent's accuracy is the same as for* $n = 1$*, and only noise agents comment visibly.*

Proof. For any bounded private signal distribution we can define $\overline{p}$ as the most extreme private belief in favour of $\theta = 1$, and $\underline{p}$ in favour of $\theta = 0$. The social belief of the first agent in the game is 0.5, so the most extreme overall beliefs they can form are also $\overline{p}$ and $\underline{p}$. If we choose $\overline{c}$ such that $1 - \frac{1}{1+\overline{c}} < \underline{p}$ and $\frac{1}{1+\overline{c}} > \overline{p}$, any social agent with adjusted index 1 will choose an invisible action with probability 1. The first agent to act visibly will therefore be a noise type, with probability 1. Given this, agents will understand that the first action carries no information, and so agents with adjusted index 2 will also have social belief 0.5 and the same reasoning will hold for them. Similarly, all agents know that in order for any social agent to act visibly they must have a social belief that is stronger than 0.5 in favour of some state, but for every adjusted index agents will know that all predecessors must have been noise agents. Thus with probability one no social agents ever act. □

However, I consider this quite a strong assumption on the timidity distribution (that there is exactly *zero* probability of an agent being confident beyond a certain point), particularly if the signal structure is not 'very' bounded, i.e. such that the convex hull of the support is some small interval around 0.5. Furthermore, the case of unbounded signals and a network topology satisfying expanding observations is a very important one in such models, since they are jointly sufficient for complete learning in the purely Bayesian case; that timidity alone does not prevent learning also provides a contrast to what we shall see as we introduce noise agents into the model. In at least this prominent instance therefore timidity alone is anodyne.

Noise agents, however, are enough to damage learning alone. In their presence, we cannot have complete learning in any network with neighbourhoods of bounded size even in the absence of timidity. The finite bound M on neighbourhood size and the presence of noise agents ensure that any agent could always observe exclusively noise agents. Since types are unobservable, and every agent is always aware that they may have observed a sample of entirely noise agents, their social beliefs will always reflect this possibility. This bounds the social beliefs one can form strictly away from 0 and 1, which implies incomplete learning. Specifically, we can bound the ex-ante probability an agent matches the state $\alpha_n := \mathbb{P}(x_n = \theta)$ as in Proposition 5 below.

Proposition 5 (Incomplete learning with finite neighbourhoods and noise agents)**.** *If there is some integer* M *such that* $|B(n)| \leq M$ *for any* $n \in \mathbb{N}$*, and non-zero measure of noise agents* ρ*, ex-ante accuracy* α_n *is bounded above by:*

$$\alpha_n \leq \frac{1}{2}\mathbb{G}_0\left(\frac{(1-\frac{\rho}{2})^M}{(1-\frac{\rho}{2})^M + (\frac{\rho}{2})^M}\right) + \frac{1}{2}\left(1 - \mathbb{G}_1\left(1 - \frac{(1-\frac{\rho}{2})^M}{(1-\frac{\rho}{2})^M + (\frac{\rho}{2})^M}\right)\right)$$

This accuracy, and the upper bound on social beliefs that underpins it, have the obvious comparative statics in ρ and M: increasing the fraction of noise agents in the population, ρ, reduces accuracy as each social signal observed is even less likely to reflect the private information of earlier agents; increasing M has the opposite effect, as each agent has more draws in which they could be observing some social predecessor. Beyond this, however, we shall see the impact of noise agents can vary dramatically as we

introduce timidity alongside them. When one has both noise agents and timidity, there are two major ways in which timidity can change outcomes: (1) by impacting the level of asymptotic learning, that is to say varying the asymptotic probability (or probabilities, since accuracy need not converge) with which agents match the state; (2) by leading to an overrepresentation of noise agents (where the probability with which any visibly-acting agent is a noise agent is higher than the probability with which a randomly selected agent is a noise agent ρ).[15]

4 Discussion

Finally, a few extensions merit discussion, to establish to what degree the results of this paper depend on the exact model specification I work with. Before this, however, I discuss my assumption that agents do not update their beliefs in response to observing $\tilde{n}$, and how this relates to the *Sleeping Beauty problem*. In such a model with endogenous neighbourhoods, social agents will form beliefs over their true indices given the adjusted indices they observe. They will also, in principle, update their beliefs about θ upon observing their adjusted index (I call these 'indexical beliefs'). For the main body of this article, I assumed that agents consider their adjusted indices to be uninformative about θ since, for every adjusted index $\tilde{n}$, an agent with this adjusted index will exist with probability 1 in each state of the world, and the conditional probability of receiving any adjusted index $\tilde{n}$ is zero in both states of the world.[16] This is where the *Sleeping Beauty problem* raises its head. There are two standard positions on this problem, 'halfer' and 'thirder', which disagree on whether one should update on the mere fact of finding oneself in a particular position. The fact that each adjusted index occurs with probability 1 in both states of the world establishes that for halfers, the agents should indeed not update their beliefs upon observing their adjusted index. I would argue that the fact that each adjusted index has a conditional probability of zero is sufficient for thirders to conclude the same. An alternative approach here would be to consider what beliefs agents would form with an improper prior over their own indices. I consider this in an online appendix of this article (Appendix D), and show that the results of the paper still hold with this assumption due to the presence of noise agents, with the exception of Theorem 3 (where by assumption there are no noise agents). An additional appeal of the halfer position is that it requires much less cognitive sophistication from agents, and is therefore more behaviourally plausible. From the analyst's perspective, it also makes the model more tractable.

Turning to the robustness of these results to the model specification, I note first of all that though they are convenient for concreteness, the exact assumptions I have made on the nature of the social signal

[15]Whether or not this is necessarily harmful depends on the application; in some instances one may not care how many of the agents are noise types, in others one could argue that it is even more important than what happens with learning. I argue that political debates on social media are an application in which there is clearly reason to worry about the presence of bots beyond their impact on learning, in that they can reduce the quality of online argumentation and give the impression that society is more polarised than it is. Some empirical evidence suggests that polarisation is driven by the very perception of polarisation, and that in turn is associated with greater levels of acceptance of political violence (Piazza, 2022).

[16]This follows from the fact that each possible index is equally probable to the agent (by the principle of insufficient reason), but that since there are infinitely many possible indices the probability of any is zero.

are not necessary for the unravelling results of this paper. Instead it suffices to assume that every agent observes at most $M \in \mathbb{N}$ agents with common knowledge of M, and that the private signals and subset of previous agents seen are independent of θ and n. Instead of working with a model à la Acemoglu et al. (2011), we could follow Smith and Sørensen (2008), where agents observe an unordered, anonymous sample of predecessors, or Monzón and Rapp (2014) where they do not even observe their own adjusted index. Beyond this one could consider other specifications, for example without binary x and v action sets. For my results to hold, it is important that agents cannot tell whether or not their neighbours are noise or social types. If we introduce another level of visibility, for example where agents can make their action observable to only some subset of predecessors and diminish the reward and punishment when seen to act correctly or incorrectly respectively, we would need to specify that noise agents also make this choice with some probability. The same is true if we allow agents to choose from a large set of feasible actions $x_n \in X \supset \{0, 1\}$: we must in turn assume noise agents randomise over that same set of actions X. It is of course true that the failures of learning that timidity can prevent, as per Theorem 1, become less severe the finer the set of available actions, disappearing entirely when this set is continuous. The binary setting is a convenient and simple one in which to illustrate the possible impact of visibility in any case, and extending it to finitely many actions will not similarly render Theorem 1 redundant. In a similar vein, we do not need to assume that noise agents all act visibly. If they instead act visibly with some probability, this is equivalent to nature selecting noise agents with lower probability. Adjusting noise agent behaviour in a different way, one may also be interested in a model where noise agents all choose the same action $x_n = 1$, as in a disinformation campaign to influence an election for example. I consider this possibility in Online Appendix E and find that most of my results extend. Counterintuitively, in sparse networks such bots would actually do more damage to the state they are trying to promote than to the opposing state.

5 Conclusion

That individuals have a choice whether or not to reveal their actions is of clear relevance in many situations in life involving social learning, yet despite this such models have thus far neglected to model said choice. Particularly in debate, and even more so on social media platforms, that agents may choose to keep their views to themselves is likely to play an important role in the outcome, as would the presence of noise.

Studying a model in which agents only reveal their actions when they are sufficiently confident in them, I discover that the combination of timidity and noise agents will affect learning in different ways in sparse and dense networks. In sparse networks, it can create an unravelling effect that leads to an exaggerated presence for noise agents amongst those who act visibly. With enough timidity, social learning can be completely shut down in sparse networks, and with moderate levels it will nonetheless be damaged. As a counterpoint to this, a little timidity can actually help here by silencing very uncertain agents.

In contrast, with dense networks the picture is generally positive. The unravelling that threatens sparse networks does not occur, and for bounded beliefs timidity and noise agents can remedy incomplete learning by eliminating the cascade beliefs that imply it in their absence. This factor depends on both

the presence of timidity *and* noise agents, and provides a setting in which, counterintuitively, malicious agents creating bots to harm society will inadvertently help it. More generally, we can observe that the conclusions of the sequential social learning literature should not be taken for granted with endogenous action visibility, which can interact with other features of a learning game in surprising ways.

A Useful Definitions

Definition 1 (Expanding Observations)**.** *A network topology has expanding sample sizes if for all $K \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \mathbb{Q}_{\tilde{n}}\Big(|B(\tilde{n})| < K\Big) = 0$$

Definition 2 (Antisymmetric Distributions)**.** *A pair of private belief distributions $\{\mathbb{G}_0, \mathbb{G}_1\}$ are antisymmetric if for all $r \in [0,1]$ $\mathbb{G}_0(r) = 1 - \mathbb{G}_1(1-r)$.*

Definition 3 (Complete Learning)**.** *Complete learning obtains if x_n converges to θ in probability (according to measure $\mathbb{P}_\sigma$) in all equilibria $\varsigma \in \Sigma$, i.e.*

$$\lim_{n\to\infty} \alpha_n := \lim_{n\to\infty} \mathbb{P}(x_n = \theta) = 1$$

B Useful Lemmas

In this section I state and prove lemmas that will be useful throughout the paper.

Lemma 4. *If the private belief distributions are antisymmetric, every social belief distribution is also antisymmetric. That is to say, the probability agent n forms belief $sb > 0.5$ if the state is 1 is equal to the probability they form social belief $1 - sb$ if the state is 0.*

Proof. In either state of the world θ, a realisation for the first n private signals directly implies the action choices of these first n agents, and a distribution over all possible choices of visibility (implied by the timidity distribution). If we take these private beliefs, and replace each with the difference between that belief and 1, we will exactly reverse the action choices, but change in no way the probability with which any selection of these agents acts visibly. Observe that with antisymmetric private signals, the probability with which the private beliefs take one profile of realisations in state $\theta = 1$ is identical to the probability they take the difference between those beliefs and 1 if $\theta = 0$. Thus the probability of any sequence of actions and visibility choices in $\theta = 1$ is exactly the same as the probability of the opposite sequence of actions (swapping every 0 with a 1 and vice versa) and the same visibility choices in $\theta = 0$. Hence a social signal which produces social belief sb is exactly as likely in state 1 as one that produces $1 - sb$ in state 0. This means the social belief distributions are antisymmetric whenever private belief distributions are. □

The following lemma justifies my speaking of social signals with a higher 'p' being more Blackwell informative in graphs such as Figure 7.

Lemma 5. *Games with antisymmetric private signal structures are games in which observing only $\tilde{x}_{\tilde{n}}$ is more Blackwell informative if and only if the unconditional probability with which it matches the state is higher.*

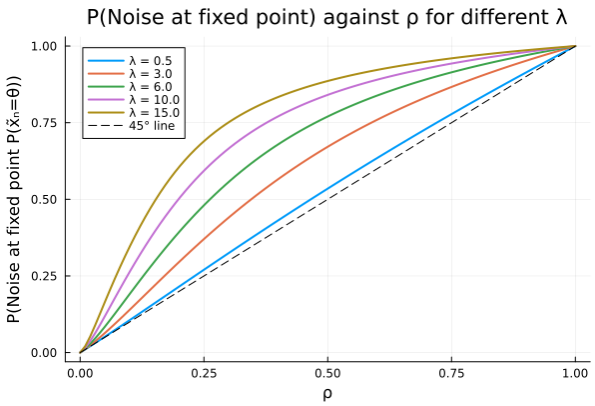
Proof. According to Lemma 4 the social belief distributions are antisymmetric in this case. If one is observing only one action there are only two beliefs that can be generated. By antisymmetry, whichever belief sb is induced by the observation of $\tilde{x}_{\tilde{n}} = 1$, it must be equally likely to form social belief $1 - sb$ when $\theta = 0$ as sb when $\theta = 1$. Hence in each state of the world these are the two possible beliefs, and that belief which supports the true state is necessarily most likely, occurring with the same probability in each state. The unconditional probability of matching the state is then simply the conditional probability of doing so in each state, they are the same. □

Lemma 6 (Monotonic Noise with Antisymmetric Private Beliefs)**.** *Suppose the private signals induce antisymmetric private belief distributions. If we take two social signals, SS_1 and SS_2, in which SS_1 Blackwell dominates SS_2, then the probability with which the visible agent who comments upon observing the signal is a noise agent is lower for SS_1 than SS_2*

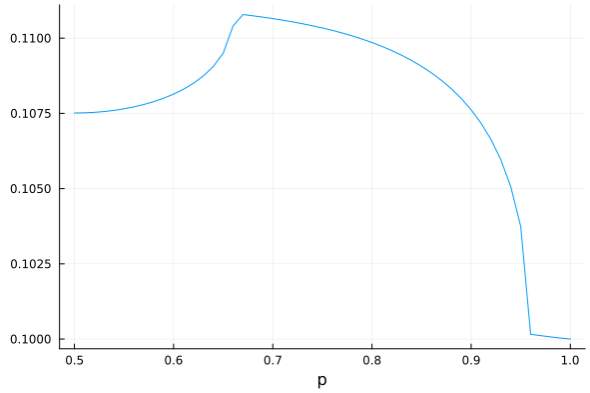
Proof. If SS_1 dominates SS_2, then the information set of SS_1 and a private signal Blackwell dominates the information set of SS_2 and a private signal by Blackwell (1951, Theorem 12). This means the unconditional (on the state) distribution of posteriors induced by SS_1 and the private signal is a mean-preserving spread of the set induced by SS_2 (Blackwell, 1953, Theorem 2).[17] Therefore there will be a higher unconditional probability of those beliefs being in the extremes demarcated by the thresholds in Lemma 2 for SS_1 than SS_2. Using Lemma 4, we know that the fact that the private belief distributions are antisymmetric means that every possible social belief $sb > 0.5$ occurs with the same conditional probability when $\theta = 1$ as $1 - sb$ does when $\theta = 0$. The probability of a private signal that, in $\theta = 1$ and combined with sb or $1 - sb$, induces an overall belief within the invisible action threshold, is the same as the probability of one that in $\theta = 0$ induces an invisible belief when combined with $1 - sb$ and sb respectively. Therefore, the conditional probabilities of an invisible signal induced by SS_1 in $\theta = 0$ and $\theta = 1$ are equal (and the same is true for SS_2). For a given agent, these facts imply probabilities $\mathbb{P}(Silent|SS_1)$ and $\mathbb{P}(Silent|SS_2)$ where the former is larger than the latter. The probability the next visible agent will be a noise agent is:

$$\begin{aligned}
&\rho + (1-\rho) \times \mathbb{P}(Silent|SS) \times \rho + \Big((1-\rho) \times \mathbb{P}(Silent|SS)\Big)^2 \times \rho + ... \\
&= \rho \sum_{j=0}^{\infty} \Big((1-\rho) \times \mathbb{P}(Silent|SS)\Big)^j \\
&= \frac{\rho}{1 - \Big((1-\rho) \times \mathbb{P}(Silent|SS)\Big)}
\end{aligned} \tag{B.1}$$

[17] This theorem states that Blackwell dominance implies a mean-preserving spread on the experimental outcome space. To apply this to posterior beliefs, simply define a new set of Blackwell experiments that map directly to the posterior belief space (applying the original probability mapping, and then Bayesian belief updating).



(a) Antisymmetric private signal pdf $\{2(1-s), 2s\}$ and Shifted Poisson timidity ($t-1$ distributed according to a Poisson distribution with mean λ).



(b) This plots the probability with which the next visible-acting agent is a noise type, if their social signal is a Bernoulli trial with success probability p, and the parameters are as set out below Lemma 6. It is clearly non-monotonic.

Figure 8: The right panel shows that without antisymmetric private belief distributions the probability with which an agent is a noise agent is not necessarily monotonic. The left panel shows a case with antisymmetric private beliefs, in which the presence of this monotonicity gives that for every ρ, asymptotic noise is increasing in timidity, and for all timidity distributions it is also increasing in ρ.

Since $\mathbb{P}(Silent|SS_1) \geq \mathbb{P}(Silent|SS_2)$, the above expression is smaller for SS_1 than SS_2. This completes the proof. □

Note that this is not true for more general private signal distributions. For example, if we have a line network with $\rho = 0.1$, a t-distribution that assigns 50% probability to 2.5 and 3.5 each, and normally distributed signals $N(-0.15, 0.1^2)$ and $N(-0.15, 0.6^2)$ in states 0 and 1 respectively, then the probability with which an agent is a noise agent is non-monotonic as seen in Figure 8b.

A consequence of Lemma 6 is that if the private belief distributions are antisymmetric, we can compute an upper bound for the probability with which a visibly acting agent is a noise agent (as opposed to $\underline{\rho}_\infty$ in the text, which is a lower bound) for any network topology, but for a given timidity distribution and private signal structure, by computing this probability for $p = 0.5$ (when the social signal is just noise).

Corollary 3.1 (Corollary to Lemma 6)**.** *If the private belief structure is antisymmetric, the probability with which any visibly acting agent $\tilde{n}$ is a noise agent can be bounded above by:*

$$\mathbb{P}(\tilde{\tau}_{\tilde{n}} = N) \leq \frac{\rho}{1-(1-\rho)\int_1^\infty \mathbb{G}_1(\frac{t}{t+1}) - \mathbb{G}_1(\frac{1}{t+1})d\Delta(t)}$$

Proof. As explained above, Lemma 6 gives us that this probability is decreasing in the informativeness of the social signal, and thus at its maximum when the social signal is pure noise. When this is so, the social belief it produces is 0.5. Given this, the overall belief of the agent is simply their private belief and we can directly apply Lemma 2 to the private belief. This gives us the visible-action thresholds, and

agents will choose to act invisibly if they observe a private signal within this range for a given t. Then we need only integrate across t, and substitute this probability into Equation B.1. That the distributions are antisymmetric means that it does not matter which distribution we use (that for $\theta = 0$ or that for $\theta = 1$). □

For example, if all agents have the same timidity parameter and the density functions are $\{2(1-s), 2s\}$, the upper bound probability with which an agent is a noise agent can be computed as follows:

$$(\frac{t}{t+1})^2 - (\frac{1}{t+1})^2 = \frac{t-1}{t+1}$$

Substituting this into our expression from Corollary 3.1

$$\mathbb{P}(\tilde{\tau}_{\tilde{n}} = N) \leq \frac{\rho}{1-(1-\rho)\frac{t-1}{t+1}}$$

If $t = 1$ this is simply ρ, as one would hope. If $t = 2$ it becomes $\frac{3\rho}{2+\rho}$, and as $t \to \infty$ it converges to 1 for any ρ.

The following lemma will be useful in the proof of Theorem 3.

Lemma 7 (Strong beliefs imply high accuracy)**.** *Suppose that we refer to the event that an agent n acting at visible history h^v forms belief weakly stronger than $\frac{t}{t+1}$ that the true state is θ for either $\theta \in \{0, 1\}$ as 'Strong'. Then it follows that $\mathbb{P}(x_n = \theta | Strong) \geq t/t + 1 > \mathbb{P}(x_n = \theta | \neg Strong)$.*

Proof. This statement is effectively an elaborate restatement of the definition of a belief, where in this paper all beliefs are defined as probabilities the agent assigns to the event $\theta = 1$. An agent's *'Belief that $\theta = 1$'* $= \mathbb{P}(\theta = 1|Belief)$.[18] Similarly their *'Belief that $\theta = 0$'* $= \mathbb{P}(\theta = 0|Belief)$. Let us define *Belief* as the agent's belief that the true state is $\theta = 1$. Since an agent will choose x_n to be whichever state they consider most likely, we have that $\mathbb{P}(x_n = \theta|Belief) = \max\{\mathbb{P}(\theta = 0|Belief), \mathbb{P}(\theta = 1|Belief)\} \geq 0.5$.

Using the definition of '*Strong*', we can observe that $\max\{\mathbb{P}(\theta = 0|Belief, Strong), \mathbb{P}(\theta = 1|Belief, Strong)\} \geq \frac{t}{t+1}$. Thus it follows that $\mathbb{P}(x_n = \theta|Strong) \geq \frac{t}{t+1}$. Similarly conditioning on the event $\neg Strong$ gives us that $\max\{\mathbb{P}(\theta = 0|Belief, \neg Strong), \mathbb{P}(\theta = 1|Belief, \neg Strong)\} < \frac{t}{t+1}$, which gives the second half of our inequality. □

Lemma 8. *Suppose we are at visible history h^v, where the action of agent $\tilde{n}$ can be observed; the following four statements concerning the success probabilities of agents $\tilde{n}(h^v)$ and $n(h^v)$ are true:*

1. $\mathbb{P}(x_{n(h^v)} = \theta) \geq \mathbb{P}(\tilde{x}_{\tilde{n}} = \theta)$

[18] A similar tautology is also used to prove Lemma A1 in Acemoglu et al. (2011)

2. $\mathbb{P}(\tilde{x}_{\tilde{n}(h^v)} = \theta | \tilde{\tau}_{\tilde{n}(h^v)} = S) \geq \mathbb{P}(x_{n(h^v)} = \theta | \tau_{n(h^v)} = S)$

3. $\mathbb{P}(\tilde{x}_{\tilde{n}(h^v)} = \theta | \tilde{\tau}_{\tilde{n}(h^v)} = S) \geq \mathbb{P}(x_{n(h^v)} = \theta)$

4. $\mathbb{P}(\tilde{x}_{\tilde{n}(h^v)} = \theta | \tilde{\tau}_{\tilde{n}(h^v)} = S) \geq \mathbb{P}(\tilde{x}_{\tilde{n}} = \theta)$

Proof. Point 1 is information monotonicity and follows from basic improvement reasoning (Acemoglu et al., 2011, Lemma A2), it says that an observer of the $\tilde{n}$th visible agent must do better than them (this says and assumes nothing about whether or not the observer then chooses to act visibly). Point 2 follows from Lemma 7. It states that conditional on choosing to act (or conditional on the agent that chooses to act being social, which is equivalent) a social agent must have a higher accuracy than one (another social agent that is) who does not necessarily choose to act. Point 3 observes that since a social agent must do better than a noise agent at a given visible history (which is just a specific information set before observing private signals), and since point 2 implies that a social visibly-acting agent following $\tilde{n}$ must do better than social agents at the same information set not necessarily acting visibly, then the accuracy of the agent acting at that visible history conditional on their being social must be higher than agents of arbitrary type and visibility. Point 4 then follows from combining points 1 and 3, and tells us that the curve in diagrams such as Figure 7 (that plot the success probability of a visible agent in an immediate predecessor conditional on their being social) must dominate the 45-degree line i.e. that information monotonicity holds conditional on no noise agents being drawn (which makes sense given Theorem 3, although they are not identical situations since in Theorem 3 all agents know that no agent is a noise agent). Hence whether or not this curve is monotonic, it will converge to 1 as the accuracy of the social signal increases to 1.[19] □

C Omitted Proofs

C.1 Decision Rule Results

Proof of Lemma 2. To maximise their expected utility, an agent will choose $v_n = 1$ if they believe the true state is $\theta = 1$ either with probability strictly greater than $\frac{t}{1+t}$ or strictly less than $\frac{1}{1+t}$.[20] Suppose without loss that $\mathbb{P}(\theta = 1) \geq \frac{1}{2}$, a symmetric argument will apply otherwise. We can derive the desired right-hand side inequality as follows.

[19] It is easy to speak of the accuracy of a given agent converging to one as the accuracy of the social signal increases to 1 in a line network with an antisymmetric private belief structure, as then the accuracy of the social signal is well defined as the probability with which it (a symmetric Bernoulli trial) matches the state. Beyond this, however, the statement intuitively holds for any signal structure and social information set. To formalise this, one could consider the maximally and minimally informative symmetric Bernoulli trials that an agent confronted with the decision problem of this paper would disprefer and prefer respectively to a given social signal A, with success probabilities $\underline{p}$ and $\overline{p}$. If then we pick a series of social signals such that both of these probabilities are converging to 1, the probability a social agent acting at this visible history will correctly match the state will then also converge to 1.

[20] They are indifferent between acting visibly or invisibly when this probability takes exactly this value. For convenience, I assume they break ties in favour of invisibility, though with absolutely continuous signal or timidity distributions it does not matter.

An agent will choose $v_n = 1$ upon observing information set I_n iff.:

$$\mathbb{P}(\theta = 1|I_n) > \frac{t}{1+t}$$

The first steps proving this statement are the same as for Lemma 1, which is simply Acemoglu et al. (2011, Proposition 2), except that we have the conditions that $\mathbb{P}(\theta = 1|I_n) > \frac{t}{1+t}$ rather than $\mathbb{P}(\theta = 1|I_n) > \frac{1}{2}$. Following these steps one arrives at:

$$d\mathbb{P}_\sigma(I_n|\theta = 1) > d\mathbb{P}_\sigma(I_n|\theta = 0)t$$

Using the fact that $d\mathbb{P}_\sigma(I_n|\theta = j) = d\mathbb{P}_\sigma(p_n|\theta = j)\mathbb{P}_\sigma(B(n)|\theta = j)$, this becomes:

$$\begin{aligned}&\Big[d\mathbb{P}_\sigma(p_n|\theta = 1) + d\mathbb{P}_\sigma(p_n|\theta = 0)\Big]\mathbb{P}_\sigma(B(n)|\theta = 1)\\ &> d\mathbb{P}_\sigma(p_n|\theta = 0)\mathbb{P}(B(n)|\theta = 1) + d\mathbb{P}_\sigma(p_n|\theta = 0)\mathbb{P}_\sigma(B(n)|\theta = 0)t\end{aligned}$$

Now we divide by $\Big[d\mathbb{P}_\sigma(p_n|\theta = 1) + d\mathbb{P}_\sigma(p_n|\theta = 0)\Big]$, apply Bayes' Rule (using again that the common prior is $\frac{1}{2}$), divide by $\mathbb{P}_\sigma(B(n)|\theta = 1) + \mathbb{P}_\sigma(B(n)|\theta = 0)$ before using Bayes' rule again to get:

$$\mathbb{P}_\sigma(\theta = 1|B(n)) > d\mathbb{P}_\sigma(\theta = 0|p_n)\mathbb{P}_\sigma(\theta = 1|B(n)) + d\mathbb{P}_\sigma(\theta = 0|p_n)\mathbb{P}_\sigma(\theta = 0|B(n))t$$

With further algebra this yields:

$$S_n > 1 + \frac{t-1}{t}d\mathbb{P}_\sigma(\theta = 1|p_n)d\mathbb{P}_\sigma(\theta = 1|B(n))$$

□

C.2 Proofs for Section 3.1

Proof of Theorem 1. This proof uses standard tools from Smith and Sørensen (2000). Without loss, suppose the true state is $\theta = 1$, and define the likelihood ratio

$$\ell_{\tilde{n}} = \frac{1 - sb_{\tilde{n}}}{sb_{\tilde{n}}}.$$

Let $\phi(sb, \theta)$ denote, in state θ, the probability that the first visibly-acting agent at a history with social belief sb chooses action 1. For each part of the statement, three steps provide the proof:

1. By the Martingale Convergence Theorem (Breiman, 1968, Theorem 5.14), ℓ almost surely does not converge to infinity (*Incorrect Learning* a.s. does not obtain).

2. By Smith and Sørensen (2000, Theorem B.1), since this is a *Markov-Martingale System* (which Smith and Sørensen describe immediately before they present this theorem), ℓ almost surely con-

verges to a stationary social belief; i.e., one such that $\phi(sb,0)=\phi(sb,1)$, and almost surely *not* to any other point.

3. Under the condition of each part, this condition is only satisfied at $\ell=0$ ($sb=1$), therefore ℓ almost surely converges to this value. This establishes that complete learning obtains. If the confidence condition fails in the second part, there are interior social beliefs such that $\phi(sb,0)=\phi(sb,1)$, and the system converges to one of these social beliefs almost surely.

It remains only to verify $\phi(sb,1)>\phi(sb,0)$ for all interior social beliefs in each part.

For a social agent with timidity t, the private belief space divides into three regions. By Lemma 2, the agent acts invisibly when $p\in(p^{**}(t),p^{*}(t))$, and visibly otherwise. By Lemma 1, the agent chooses $x_n=1$ if $p>1-sb$ and $x_n=0$ if $p<1-sb$. Since $p^{**}(t)\le 1-sb\le p^{*}(t)$, these two rules combine as follows: if $p>p^{*}(t)$ the agent is *visible and chooses* $x_n=1$; if $p\in(p^{**}(t),p^{*}(t))$ the agent is *invisible*; and if $p<p^{**}(t)$ the agent is *visible and chooses* $x_n=0$. A noise agent always acts visibly and chooses $x_n=1$ with probability $\frac{1}{2}$. By Lemma 2:

$$p^{*}(t)=\frac{t\big(1-sb\big)}{sb+t\big(1-sb\big)},\qquad p^{**}(t)=\frac{1-sb}{1+(t-1)sb}.$$

Note that p^{*} is continuous and strictly increasing in t, from $p^{*}(1)=1-sb$ to $p^{*}(t)\to 1$ as $t\to\infty$; while p^{**} is continuous and strictly decreasing, from $p^{**}(1)=1-sb$ to $p^{**}(t)\to 0$. Define the average visible-right and visible-left probabilities, integrated over the type distribution:

$$R_\theta:=\int\big[1-\mathbb{G}_\theta\big(p^{*}(t)\big)\big]d\Delta(t),\qquad L_\theta:=\int\mathbb{G}_\theta\big(p^{**}(t)\big)d\Delta(t).$$

Then

$$\phi\big(sb,\theta\big)=\frac{\rho/2+(1-\rho)R_\theta}{\rho+(1-\rho)(R_\theta+L_\theta)}. \tag{C.1}$$

Write $\phi=h(R,L)$ where $h(R,L):=\frac{\rho/2+(1-\rho)R}{\rho+(1-\rho)(R+L)}$. Since $\rho>0$, h is strictly increasing in R and strictly decreasing in L.[21] By Acemoglu et al. (2011, Lemma A1(c)), $\mathbb{G}_1$ first-order stochastically dominates $\mathbb{G}_0$ ($\mathbb{G}_1(x)\le\mathbb{G}_0(x)$ for all x, with strict inequality on the interior of the support of $\mathbb{G}_\theta$), giving pointwise in t:

$$1-\mathbb{G}_1(p^{*}(t))\ge 1-\mathbb{G}_0(p^{*}(t)),\qquad \mathbb{G}_1(p^{**}(t))\le\mathbb{G}_0(p^{**}(t)),$$

with strict inequality whenever $p^{*}(t)$ or $p^{**}(t)$ lies in the interior of the support where FOSD is strict. Integrating over t: $R_1\ge R_0$ and $L_1\le L_0$, and by the monotonicity of h, $\phi(sb,1)>\phi(sb,0)$ whenever at least one of these inequalities is strict. It suffices to verify this strictness under the hypotheses of each part.

Part 1. With unbounded private beliefs, Acemoglu et al. (2011, Lemma A1(c)) gives $\mathbb{G}_1(x)<\mathbb{G}_0(x)$

[21] Explicitly: $\frac{\partial h}{\partial R}=\frac{(1-\rho)\left[\rho/2+(1-\rho)L\right]}{D^2}>0$ and $\frac{\partial h}{\partial L}=\frac{-(1-\rho)\left[\rho/2+(1-\rho)R\right]}{D^2}<0$, where $D=\rho+(1-\rho)(R+L)$.

for all $x \in (0,1)$. Since $p^*(t), p^{**}(t) \in (0,1)$ for every $t \geq 1$ and every interior sb, the inequalities are strict for every t: $R_1 > R_0$ and $L_1 < L_0$, giving $\phi(sb,1) > \phi(sb,0)$. No interior sb is stationary.

Part 2. With bounded private beliefs supported on $[\underline{p}, \overline{p}] \subset (0,1)$, Acemoglu et al. (2011, Lemma A1(c)) gives $\mathbb{G}_1(x) < \mathbb{G}_0(x)$ for $x \in (\underline{p}, \overline{p})$, with $\mathbb{G}_1 = \mathbb{G}_0$ outside this interval. I show that for every interior sb, at least one threshold $p^*(t)$ or $p^{**}(t)$ lies in $(\underline{p}, \overline{p})$ for a positive-Δ-measure set of timidity values. Define

$$c^* := \frac{\underline{p}(1-\overline{p})}{\overline{p}(1-\underline{p})}.$$

The hypothesis $(0, c^* + \epsilon) \cup (1-\epsilon, 1) \subseteq \text{supp}(\Delta(c))$ for some $\epsilon > 0$ is used as follows.

- $sb \in (1-\overline{p}, 1-\underline{p})$*:* $p^*(1) = 1 - sb \in (\underline{p}, \overline{p})$. By continuity, $p^*(t) \in (\underline{p}, \overline{p})$ for all t in a neighbourhood of 1, i.e. for $c = 1/t$ in a neighbourhood of 1. Since $(1-\epsilon, 1) \subseteq \text{supp}(\Delta)$, these timidity values carry positive mass.

- $sb \geq 1-\underline{p}$*:* $p^*(1) = 1 - sb \leq \underline{p}$, but $p^*(t) \to 1$, so as t increases $p^*(t)$ enters $(\underline{p}, \overline{p})$. The set of timidity values with $p^*(t) \in (\underline{p}, \overline{p})$ corresponds to an open interval of c-values with left endpoint

$$c_{\min}(sb) = \frac{(1-sb)(1-\overline{p})}{\overline{p} \cdot sb},$$

  which is at most c^* (attained at $sb = 1-\underline{p}$) and tends to 0 as $sb \to 1$. Since $(0, c^* + \epsilon) \subseteq \text{supp}(\Delta)$ and $c_{\min} \leq c^*$, the interval of timidity values which produce visible and invisible action with strictly positive probability always intersects the support.

- $sb \leq 1-\overline{p}$*:* Here we can make a symmetric argument applied to $p^{**}(t)$, which enters $(\underline{p}, \overline{p})$ from above as t increases. The left endpoint of the corresponding c-interval is again at most c^* (attained at $sb = 1-\overline{p}$), so the same condition $(0, c^* + \epsilon) \subseteq \text{supp}(\Delta)$ provides the needed positive-measure interval of timidity values such that agents will act visibly and invisibly with strictly positive probability.

In each case, strict FOSD at the relevant threshold for a positive-measure set of timidity values gives $R_1 > R_0$ or $L_1 < L_0$, and hence $\phi(sb,1) > \phi(sb,0)$. No interior sb is stationary, so the social belief converges to certainty on the true state almost surely. □

C.3 Proofs for Section 3.2

Proof of Proposition 1. If there are values of t strictly greater than t_0^* in the support of the t-distribution, there is some interval $[L_0(t), U_0(t)]$ associated with each of them around 0.5. Since there is some interval $(0.5-\delta, 0.5+\delta)$ included within the support of the belief distribution, then for each t $[L_0(t), U_0(t)] \cap (0.5-\delta, 0.5+\delta)$ is also within the support of the private signal distribution.

The region $\bigcup_{t>t_0^*} \left([L_0(t), U_0(t)] \cap (0.5-\delta, 0.5+\delta)\right)$ is then a subset of the region of pairs of timidity parameters and private beliefs that produce invisible actions. Integrating across this region then produces a strictly positive probability that is strictly below λ_0. Hence $\lambda_0 > 0$ and the result follows. □

After this proposition, I note that additional conditions can guarantee that $\underline{\rho}_j > \underline{\rho}_{j-1}$ for any j. The following reasoning establishes these claims.

With a full support t-distribution, for any p within the support of the private belief distribution, there is an interval of t-values $[L_0^{-1}(p), \infty)$ or $[U_0^{-1}(p), \infty)$ (the former if p is below 0.5, the latter if above) such that the agent will choose $v_n = 0$ if they observe private signal p and have a t_n within this interval. The integral of these probabilities across the support of p is at least $\lambda_1 > 0$: (1) This is *at least* λ_1 since we have not specified the true state of the world, as λ_1 is the minimum value of the probability of a (p_n, t_n) pair that gives $v_n = 0$ in either of the two states of the world (2) This is strictly greater than zero as for any value in the support of the private signal there is a strictly positive probability the agent has a t value that implies $v_n = 0$, since we have assumed our t distribution is full support.

The fact that $\lambda_1 > 0$ implies that for any p, $U_1^{-1}(p) < U_0^{-1}(p)$ and $L_1^{-1}(p) > L_0^{-1}(p)$. This in turn implies we are integrating our private signal and t-distributions over a strictly larger region to obtain λ_2, thus obtaining a strictly larger value for λ_2: $\lambda_2 > \lambda_1$. This implies in turn that $\underline{\rho}_2 > \underline{\rho}_1$. The same reasoning holds for any $j \in \mathbb{N}$, thus $\underline{\rho}_j > \underline{\rho}_{j-1}$ for any $j \in \mathbb{N}$.

If the private beliefs distribution is full-support, then whatever t-values above t_0^* are contained within the support of $\Delta(t)$, each iteration of the reasoning must increase the interval of invisible-action private beliefs. Since the distribution of private beliefs is full support, this must imply that invisible actions are chosen with a strictly higher probability each round $\lambda_j > \lambda_{j-1}$ for all $j \in \mathbb{N}$. Thus $\underline{\rho}_j > \underline{\rho}_{j-1}$.

If the private beliefs distribution is unbounded, and the t-distribution satisfies the condition specified in the third sufficient condition, then even though mild private beliefs are not in the support there are t values in the support that can bring an agent with social belief 0.5 back into the invisible-action belief region (the t condition is defined to guarantee this). For agents with stronger social beliefs in either direction, since the $\underline{p}$ and $\overline{p}$ values are the values beyond which all private beliefs have positive support, there is necessarily also a positive probability that one of the private beliefs of sufficient strength to push the agent's belief back into the invisible action region will be selected. The conclusion follows from this as in the previous two cases.

Proof of Lemma 3. We have the relation:

$$\underline{\rho}_\infty\Big(1 - (1 - \underline{\rho}_\infty)\lambda_\infty(\rho_\infty^*)\Big) = \rho$$

λ_∞ is the minimum of two probabilities (one for each value of θ) that we have a private belief-timidity pair in the invisibility region:

$$\lambda_\infty = \min\left\{ \int_{t_\infty^*}^{\infty} \mathbb{G}_0(U_\infty(t)) - \mathbb{G}_0(L_\infty(t))d\Delta(t), \int_{t_\infty^*}^{\infty} \mathbb{G}_1(U_\infty(t)) - \mathbb{G}_1(L_\infty(t))d\Delta(t) \right\}$$

In general the left-hand side of our first relation is increasing in $\underline{\rho}$ so there is clearly a unique solution for $\underline{\rho}$, which is increasing in the right-hand side, ρ. Secondly, the left-hand side is decreasing in λ_∞, so a higher λ_∞ implies a higher $\underline{\rho}$ for the equation to balance. The comparative statics are then implied by the impact of each variable on λ_∞.

- For higher values of t, $\mathbb{G}_\theta(U_\infty(t)) - \mathbb{G}_\theta(L_\infty(t))$ necessarily takes a higher value for either $\theta \in \{0,1\}$. It follows that shifting probability mass to higher t values necessarily increases λ_∞, and thus $\underline{\rho}$. Therefore a FOSD shift in $\Delta(t)$ exacerbates the problem and increases the representation of noise agents.

- Increasing M necessarily decreases $U_\infty(t)$ and increases $L_\infty(t)$ for all levels of t, so it reduces λ_∞, and then reduces $\underline{\rho}$.

- Also, the more mass the belief distributions assign to central values (value within $U_\infty(t)$ and $L_\infty(t)$), the higher λ_∞. A mean-preserving spread in $\mathbb{G}_0$ or $\mathbb{G}_1$ reduces the representation of noise agents.

□

Proof of Proposition 2. For bounded private belief signals, Proposition 4 gives this directly. For unbounded signals, we have from the above working that any t implies a (possibly empty) interval of private signals that imply the agent receiving them will certainly act invisibly i.e. choose $v_n = 0$:

$$\mathbb{P}(\theta = 1|s_n) \in [\underbrace{1 - \frac{t_n(\frac{\rho}{2})^M}{t_n(\frac{\rho}{2})^M + (1-\frac{\rho}{2})^M}}_{L_0}, \underbrace{\frac{t_n(\frac{\rho}{2})^M}{t_n(\frac{\rho}{2})^M + (1-\frac{\rho}{2})^M}}_{U_0}]$$

This lower limit is decreasing in t and converging to 0, and the upper limit is increasing in t and converging to 1, hence the shape of the curves in Figure 3. This process of shifting the t distribution to the right actually implies that λ_0 converges to 1, before we even worry about our rounds of iterated reasoning as above ($\lambda_0 \to 1$ itself implies that $\lambda_\infty \to 1$). That this in turn implies that $\underline{\rho}_\infty \to 1$ follows immediately from the definition of $\underline{\rho}_\infty$ in Theorem 2. It remains therefore only to argue that λ_0 does in fact converge to 1 as we shift our t-distribution as described.

In computing λ_0, we integrate (for both $\theta = 0$ and $\theta = 1$, before taking the minimum) over the region within the first black curve depicted in Figure 3, and shifting the t-distribution to the right simply implies that each region of this distribution is now paired to an at least weakly wider interval of private signals, and therefore an at least weakly higher probability mass. Whatever the private belief distributions, the amount of mass assigned to $[\epsilon, 1-\epsilon]$ must be converging to 0 and ϵ converges to 0, so as the t-distribution allocates all of its mass to wider and wider intervals (converging to the complete $[0,1]$ interval), the integral must converge to 1 (as we are integrating over the entire support of both t and p_n).

Notice that we could make exactly the same argument for any λ_j, and that whilst this would not change the limit, it would make convergence faster. Integrating between the curves $[L_\infty(t), U_\infty(t)]$ would produce the fastest convergence. Since we are only establishing convergence here anyway, it is easiest to just use λ_0. Notice also that the width of intervals are decreasing in M for all values of t, so the lower M the faster the convergence. □

Proof of Proposition 3. The reasoning in this proof is very similar to that of Proposition 2. The sequence of distributions is defined such that a greater and greater proportion of the region supported by the product distribution over p_n and t is within the region in which agents certainly comment invisibly. As

before, since the mass assigned outside this region converges to 0, λ_∞ must converge to 1, which gives us that $\underline{\rho}_\infty \to 1$, given the definition of $\underline{\rho}_\infty$ in Theorem 2.

Note also that we could have used $[L_0^1(\underline{t}), U_0^1(\underline{t})]$ instead of $[L_\infty^1, U_\infty^1]$ if we do not want to have to compute $[L_\infty^1, U_\infty^1]$, though this is a stronger condition on the sequence of new private belief distribution pairs.

□

C.4 Proofs for Section 3.4

Proof of Theorem 3. As noted in Section 2, an agent with timidity t will choose to act visibly if they form belief stronger than $\frac{t}{t+1}$ that the state is θ, for any $\theta \in \{0, 1\}$. In the following argument, I refer to the event of the agent having such a strong belief as *Strong*.

- The first visibly acting agent after $\tilde{n}$, $\tilde{n}+1$, will be correct with probability $\mathbb{P}(x_{\tilde{n}+1} = \theta) = \mathbb{P}(x_{n(\tilde{n})+1} = \theta | Strong)$. In words, the probability the first visibly-acting agent after $\tilde{n}$ makes the correct choice is simply the probability that the first agent (visible or not) makes the correct choice conditional on their having a belief stronger than $\frac{t}{t+1}$ in favour of one state or the other.

- This probability must be higher than the unconditional probability that $\tilde{n}$'s immediate successor (visible or not) will be successful, since $\mathbb{P}(x_{n(\tilde{n})+1} = \theta | Strong) \geq \mathbb{P}(x_{n(\tilde{n})+1} = \theta | \neg Strong)$ by Lemma 7 and:[22]

$$\begin{aligned}\mathbb{P}(x_{n(\tilde{n})+1} = \theta) &= \mathbb{P}(x_{n(\tilde{n})+1} = \theta | Strong)\mathbb{P}(Strong) + \mathbb{P}(x_{n(\tilde{n})+1} = \theta | \neg Strong)\mathbb{P}(\neg Strong) \\ &= \mathbb{P}(x_{n(\tilde{n})+1} = \theta | Strong)\mathbb{P}(Strong) + \mathbb{P}(x_{n(\tilde{n})+1} = \theta | \neg Strong)\big(1 - \mathbb{P}(Strong)\big)\end{aligned}$$

- That $\mathbb{P}(x_{n(\tilde{n})+1} = \theta)$ must be higher than $\mathbb{P}(x_{n(\tilde{n})} = \theta)$ however, follows from the improvement principle reasoning of Acemoglu et al. (2011). It thus follows that an improvement path between any visibly-acting agents must produce a greater lower bound on the final agent's accuracy than in an analogous path in the model of Acemoglu et al. (2011).

- Thus unless after a point no agent will ever act visibly again, the argument of Acemoglu et al. (2011, Theorem 2) implies that the accuracy of visibly-acting agents must converge in probability to 1. Since Borel-Cantelli establishes that after any given history there must almost surely be another visibly-acting agent (in fact, infinitely many of them), there cannot be such a point. Hence the accuracy of visibly-acting agents converges to 1 in probability.

- Any invisibly-acting agent is also improving on these agents, and so their accuracy converges in probability to 1 as well. The probability with which an agent acts invisibly must also converge to 1, since the strength of social beliefs is converging to 1.

[22] This follows from the nature of Bayesian belief formation, though trivially it can be seen as following from Blackwell's Theorem. I.e. compare providing an agent with a signal that provides the weakest possible *Strong* belief in either direction against one that forms any belief that is $\neg$*Strong*. Since any such weak signal can be represented as a garbling of the strong signal, it must be dispreferred in expected utility for all decision problems, and give a lower probability of matching the state.

The above argument assumed a given t, but works for an arbitrary value of it. Integrating across t values establishes that it holds for any distribution of t values.

□

Proof of Proposition 5. Suppose agent n observes M others, and suppose each of these chooses action 1. Further suppose that at this point in the game social agents are informed that their actions are perfectly informative. This is the strongest possible signal an agent can observe in favour of $\theta = 1$, and thus establishes the bound, though one could go through the basic reasoning of the following argument for any social signal, since for each possible social signal there is some probability it was produced by observing only noise agents. Using Bayes' rule, the probability that upon observing all 1s one's entire neighbourhood is comprised of 1-noise agents, we can compute that:

$$\mathbb{P}(\theta = 1 | M \text{ 1's observed}) = \frac{(1-\frac{\rho}{2})^M}{(1-\frac{\rho}{2})^M + (\frac{\rho}{2})^M}$$

Therefore if the true state is $\theta = 1$, our accuracy must be at most $1 - \mathbb{G}_1(1 - \frac{(1-\frac{\rho}{2})^M}{(1-\frac{\rho}{2})^M+(\frac{\rho}{2})^M})$ (using Lemma 1 here). And if the state is $\theta = 0$, the accuracy is at most $\mathbb{G}_0(\frac{(1-\frac{\rho}{2})^M}{(1-\frac{\rho}{2})^M+(\frac{\rho}{2})^M})$. Thus the ex-ante accuracy is:

$$\alpha_n \leq \frac{1}{2}\mathbb{G}_0\left(\frac{(1-\frac{\rho}{2})^M}{(1-\frac{\rho}{2})^M + (\frac{\rho}{2})^M}\right) + \frac{1}{2}\left(1 - \mathbb{G}_1\left(1 - \frac{(1-\frac{\rho}{2})^M}{(1-\frac{\rho}{2})^M + (\frac{\rho}{2})^M}\right)\right)$$

□

References

Acemoglu, D., M. A. Dahleh, I. Lobel, and A. Ozdaglar (2011): "Bayesian learning in social networks," *The Review of Economic Studies*, 78, 1201–1236.

Banerjee, A. V. (1992): "A Simple Model of Herd Behavior," *Quarterly Journal of Economics*.

Bikhchandani, S., D. Hirshleifer, and I. Welch (1992): "A Theory of Fads , Fashion , Custom , and Cultural Change as Informational Cascades," *Journal of Political Economy*, 100, 992–1026.

Blackwell, D. (1951): "Comparisons of experiments," *Berkeley Symp. on Math. Statist. and Prob.*, 93–102.

——— (1953): "Equivalent comparisons of experiments," *The annals of mathematical statistics*, 265–272.

Bohren, J. A. and D. N. Hauser (2021): "Learning With Heterogeneous Misspecified Models: Characterization and Robustness," *Econometrica*, 89, 3025–3077.

Breiman, L. (1968): *Probability*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104).

Callander, S. and J. Hörner (2009): "The wisdom of the minority," *Journal of Economic Theory*, 144, 1421–1439.e2.

Cremin, J. W. (2025): "Too Much Information & The Death of Consensus," Working paper or preprint.

Elga, A. (2000): "Self-locating belief and the Sleeping Beauty problem," *Analysis*, 60, 143–147.

Goeree, J. K., T. R. Palfrey, and B. W. Rogers (2006): "Social learning with private and common values," *Economic theory*, 28, 245–264.

Guarino, A., H. Harmgart, and S. Huck (2011): "Aggregate information cascades," *Games and Economic Behavior*, 73, 167–185.

Guarino, A. and P. Jehiel (2013): "Social learning with coarse inference," *American Economic Journal: Microeconomics*, 5, 147–174.

Herrera, H. and J. Hörner (2013): "Biased social learning," *Games and Economic Behavior*, 80, 131–146.

Janda, P. (2024): "Doppelgänger Changes the Game," *Episteme*, 21, 1182–1207.

Kierland, B. and B. Monton (2005): "Minimizing inaccuracy for self-locating beliefs," *Philosophy and Phenomenological Research*, 70, 384–395.

Lewis, D. (2001): "Sleeping beauty: reply to Elga," *Analysis*, 61, 171–176.

Lobel, I. and E. Sadler (2015): "Information diffusion in networks through social learning," *Theoretical Economics*.

——— (2016): "Preferences, Homophily and Social Learning," *Operations Research*.

Lomys, N. (2020): "Collective search in networks," *Available at SSRN 3197244*.

Monzón, I. and M. Rapp (2014): "Observational learning with position uncertainty," *Journal of Economic Theory*, 154, 375–402.

Piazza, J. A. (2022): "Political polarization and political violence," *Available at SSRN 4156980*.

Piccione, M. and A. Rubinstein (1997): "On the interpretation of decision problems with imperfect recall," *Games and Economic Behavior*, 20, 3–24.

Ross, J. (2010): "Sleeping Beauty, countable additivity, and rational dilemmas," *Philosophical Review*, 119, 411–447.

Smith, L. and P. Sørensen (2000): "Pathological Outcomes of Observational Learning," *Econometrica*, 68, 371–398.

——— (2008): "Rational social learning with random sampling," Tech. rep., working paper.

Song, Y. (2016): "Social learning with endogenous observation," *Journal of Economic Theory*, 166, 324–333.

Winkler, P. (2017): "The sleeping beauty controversy," *The American Mathematical Monthly*, 124, 579–587.

D Online Appendix: Indexical Beliefs

Upon observing their adjusted index, ignoring other information for the moment, a Bayesian agent should update their belief about the state θ. As I have mentioned, how exactly they go about doing this depends on one's stance on the *Sleeping Beauty problem*, or more precisely the *Duplicating Sleeping Beauty problem*, a variant. In the Sleeping Beauty problem (Elga, 2000), a fair coin is tossed and an individual is awoken once (on Monday) if it lands heads and twice (on Monday and then Tuesday) if it lands tails. They are always put to sleep with a drug that makes them forget previous awakenings, so all awakenings are indistinguishable, and are asked upon awakening for their credence that the coin landed heads. This problem has produced a very large literature (for which Winkler (2017) provides an excellent review), in which there are two standard positions: *halfer* ($\mathbb{P}(Heads|Awake) = \frac{1}{2}$ à la Elga (2000)) and *thirder* ($\mathbb{P}(Heads|Awake) = \frac{1}{3}$ à la Lewis (2001)). The majority of epistemologists support the thirder position, but neither has definitively 'won.' In the Duplicating Sleeping Beauty problem, instead of being awakened, Sleeping Beauty is cloned, and this clone awakened in her place. Some philosophers (Janda, 2024; Kierland and Monton, 2005) argue that even if one is a thirder in the original problem, one should be a halfer in the duplicating variant. In my model, arriving with a given adjusted index, an agent's true index is analogous to the day of the week, and the state of the world θ is analogous to the coin toss.

If one is a halfer in the Sleeping Beauty problem, or accepts the arguments that one should be in the duplicating variant (as I myself do, incidentally), then my assumption that agents do not adjust their beliefs in response to learning their adjusted index is correct. This is as the presence of noise agents means that every possible adjusted index $\tilde{n} \in \mathbb{N}$ is assigned to some agent with probability 1 in each state of the world. Applying the Generalised Halfer Principle of Ross (2010)[23], one can see that agents should not update their beliefs about the state of the world, since arriving at an information set communicates no information when the probability of doing so is the same in both states of the world.

What if, however, you are a thirder, and do not accept that anything changes in the duplicating variant of the problem? Since the common prior is $\frac{1}{2}$, the Bayesian posterior for a thirder is:

$$\mathbb{P}(\theta = 1|\tilde{n}) = \frac{\mathbb{P}(\tilde{n}|\theta = 1)}{\mathbb{P}(\tilde{n}|\theta = 0) + \mathbb{P}(\tilde{n}|\theta = 1)}$$

Given that the prior over n is uniform by the principle of insufficient reason however, this is equal to $0/(0+0)$ (the probability of being 'assigned' any $n \in \mathbb{N}$ is zero, and thus the conditional probability of having any $\tilde{n} \in \mathbb{N}$ is also zero). My assumption in the main article that agents do not update their belief upon observing $\tilde{n}$ can be defended as the sensible response to this, but an alternative approach is to endow them with an improper prior.

In this appendix I show that proceeding thus, we can nonetheless bound agents' 'indexical' beliefs away

[23] Ross speaks of 'objective chances', and here we must replace these with equilibrium probabilities, but otherwise one can apply this principle directly.

from 0 and 1, thanks to the presence of noise agents. This fact ensures that the unravelling results all still hold. Furthermore, I shall explain that the material in Section 3.3 is unaffected, and that Theorem 1 also stands with this alternative assumption. The only result whose proof does not carry forward is Theorem 3, since this benchmark result specifically considers the case in which we have no noise agents ($\rho = 0$).[24] It is worth noting that in the special case where private beliefs are antisymmetric (c.f. Definition 2), the adjusted indices $\tilde{n}$ carry no information anyway, and this approach and that assumed in the main paper are identical. Hence the figures and examples in which I consider antisymmetric private beliefs (Examples 1 and 2 both assume antisymmetric private belief distributions), are also unaffected anyway.

Imagine the agent has an improper prior, assigning probability $\epsilon > 0$ to each possible $n \in \mathbb{N}$. If their adjusted index is $\tilde{n} = 1$, they consider the hypotheses: $\{n = 1\} \cap \{\theta = 0\}$, $\{n = 2\} \cap \{\theta = 0\}$, ..., and $\{n = 1\} \cap \{\theta = 1\}$, $\{n = 2\} \cap \{\theta = 1\}$, ... This is because if you observe that no agent has visibly acted before you, this could simply be because you are the first agent to arrive, but it could also be that you are the 100th and that all 99 of your predecessors chose to act invisibly. The more agents are required to have acted invisibly, the more unlikely a hypothesis, but a Bayesian with an improper prior will consider all such possible explanations.

The probability of having $\tilde{n} = 1$ conditional on $\theta = 0$ is then, for example:

$$\begin{aligned}\mathbb{P}(\tilde{n} = 1|\theta = 0) = &\frac{1}{2} \times \epsilon + \frac{1}{2} \times \epsilon \times \mathbb{P}(v_1 = 0|\theta = 0) \\ &+ ... + \frac{1}{2} \times \epsilon \times \mathbb{P}(v_1 = 0|\theta = 0) \times ... \times \mathbb{P}(v_{k-1} = 0|\theta = 0) \\ &+ ...\end{aligned}$$

Suppose we have upper and lower bounds for the probabilities $\{\mathbb{P}(v_k = 0|\theta = 0) : k \in \mathbb{N}\}$. Call these $\underline{p}$ and $\overline{p}$. It then follows that:

$$\begin{aligned}\mathbb{P}(\tilde{n} = 1|\theta = 0) &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}\overline{p} + ... + \frac{\epsilon}{2}\overline{p}^{k-1} + ... = \frac{\epsilon}{2}\frac{1}{1-\overline{p}} \\ \text{and } \mathbb{P}(\tilde{n} = 1|\theta = 0) &\geq \frac{\epsilon}{2}\frac{1}{1-\underline{p}}\end{aligned}$$

This line of reasoning applies equally for probabilities conditional on $\theta = 1$ as $\theta = 0$, and indeed with infinitely many agents (where there will almost surely be infinitely many visibly-acting agents, thanks to the presence of noise) identical bounds can be derived for any $\tilde{n}$ (not just $\tilde{n} = 1$). Thus we can set out the following upper bound:

$$\mathbb{P}(\theta = 1|\tilde{n}) = \frac{1}{1 + \frac{\mathbb{P}(\tilde{n}|\theta=0)}{\mathbb{P}(\tilde{n}|\theta=1)}} \leq \frac{1}{1 + \frac{1-\overline{p}}{1-\underline{p}}}$$

Since noise agents always act visibly, a candidate for $\overline{p}$ is $1 - \rho$. Simply taking $\underline{p} = 0$ we can then see

[24] In such a case, one encounters the absentminded driver paradox of Piccione and Rubinstein (1997), as well as the Sleeping Beauty problem in epistemology (Elga, 2000). The question of how one should address these issues, particularly in models of social learning, is the subject of my ongoing research.

that:

$$\mathbb{P}(\theta = 1|\tilde{n}) = \frac{1}{1 + \frac{\mathbb{P}(\tilde{n}|\theta=0)}{\mathbb{P}(\tilde{n}|\theta=1)}} \leq \frac{1}{1+\rho}$$

Alternatively, in any given equilibrium we can define λ_∞ as in the main text to provide a lower bound on the probability with which any social agent acts invisibly. $(1-\rho)\lambda_\infty$ then serves as a candidate for $\underline{p}$. This gives a tighter bound:

$$\mathbb{P}(\theta = 1|\tilde{n}) \leq \frac{1-(1-\rho)\lambda_\infty}{1-(1-\rho)\lambda_\infty+\rho}$$

The tighter the bound, the lower t-values will be necessary to establish that unravelling takes place. However, for the sake of this appendix we need simply establish that we can bound indexical beliefs away from 0 and 1, so let us use the simpler one $\frac{1}{1+\rho}$. We could have run through the same steps with $\mathbb{P}(\theta = 0|\tilde{n})$, so since $\mathbb{P}(\theta = 1|\tilde{n}) = 1 - \mathbb{P}(\theta = 0|\tilde{n})$ we have also a lower bound for $\mathbb{P}(\theta = 1|\tilde{n}) = \frac{1+\rho}{1+\rho} - \frac{1}{1+\rho} = \frac{\rho}{1+\rho}$. Hence $\mathbb{P}(\theta = 1|\tilde{n}) \in [\frac{\rho}{1+\rho}, \frac{1}{1+\rho}]$.

D.1 Which results still hold?

As I have mentioned, the only result that does not carry over is Theorem 3.[25] Proposition 4 still holds since even with the more extreme beliefs that incorporating indexical information may produce, we will still be able to bound the strongest beliefs agents can form away from zero and one. Given this, as is argued in the proof of this result, a small enough upper bound on confidence $\overline{c}$ will ensure no social agents ever comment visibly. For Proposition 5, the exact expression no longer applies, though we could provide another straightforwardly by asking what accuracy agents would obtain if in state $\theta = 0$ they happened to have the most extreme indexical belief in favour of 0, and vice versa in $\theta = 1$.

All the results in Section 3.2, on unravelling, follow from the fact that we can bound social beliefs away from one and zero, generating a central interval of private beliefs that certainly induce invisible actions for high enough t. This line of reasoning remains valid with agents who have an improper indexical prior, since the prior beliefs are bounded away from zero and one. Hence Theorem 2 and Propositions 1 and 3 still hold, and there is no need to even adjust the expressions contained within. Corollary 2.1 is an updated version of Proposition 5 using $\underline{\rho}_\infty$, so the expression would need to be adjusted similarly. The qualitative point that one can find a tighter bound that reflects unravelling however, is unchanged. Similarly Proposition 2 only changes in that the accuracy value to which agents converge will depend on their adjusted index (as different adjusted indices imply different indexical beliefs), instead of converging to $\frac{1}{2}\mathbb{G}_0(0.5) + \frac{1}{2}\big(1 - \mathbb{G}_1(0.5)\big)$ for all agents.

The countervailing effects discussed in Section 3.3 (Sparse Learning), do not depend on indexical beliefs, and indeed since the examples and figures I use in this section feature antisymmetric private beliefs, none of them change if we consider agents with improper indexical priors.

In Section 3.1, I discuss learning in dense networks. Theorem 1 depends on the martingale convergence theorem, and the required application of this does not change here. It will still be the case that the social

[25] Even if one is a halfer, the fact that all information sets are reached with probability 1 in both states of the world uses the existence of noise agents.

belief of agents will be a martingale that will converge to 1 on the true state, though agents will now be combining this social belief with both a private signal *and* their indexical beliefs. Much as private beliefs are eventually dominated by a social belief that is converging to 1, indexical beliefs become insignificant also (since again they are crucially bounded away from 1 and 0). Point 2 of this theorem leverages the fact that with bounded beliefs, we can remove any cascade beliefs with enough timidity; this is also unchanged by the improper prior assumption.

Therefore, the results of the paper do not crucially depend on the fashion in which agents treat indexical information. Assuming they operate with an improper prior may increase the amount of timidity needed to produce unravelling, but the qualitative argument of the paper remains substantially unaffected.

E One-Sided Noise

In this paper I have assumed noise agents randomise uniformly between the two actions, but what, one might ask, if they were to all choose $x = 1$?[26,27] Perhaps they represent a targeted disinformation campaign by a hostile state during an election?

In this model, such agents would successfully prevent learning, but would have an exaggeratedly negative impact on the very state they were trying to promote ($\theta = 1$) in sparse networks. In dense networks they help learning in a smaller set of circumstances than in the main model (c.f. Proposition 7 below). This sparse network result obtains since upon observing a neighbourhood in which all choose $x = 1$, an agent will account for the possibility that they are all noise agents, and form a social belief no stronger than $\frac{(1-\rho)^M}{(1-\rho)^M+\rho^M}$. (In the main paper the equivalent expression replaces the ρ here with $\rho/2$.) However, no such bound holds for beliefs that $\theta = 0$. We could then ask, what private beliefs will certainly **not** induce $\{x_n = 1, v_n = 1\}$? We can then rehearse much the same unravelling argument as in the main article, except applying it uniquely to the agents visibly choosing action 1. More agents with beliefs above $\frac{1}{2}$ will choose to act invisibly, which will reduce the maximum strength of social beliefs in favour of $\theta = 1$, which will cause more such agents to act invisibly, which will lower the bound, and so on and so forth.

E.0.1 Unravelling with One-Sided Noise

Let us observe that for any private belief in the following interval, agents will not choose $\{x_n = 1, v_n = 1\}$:

$$\mathbb{P}(\theta = 1|s_n) \in [0, \underbrace{\frac{t_n\rho^M}{t_n\rho^M + (1-\rho)^M}}_{U_0^1}]$$

Following the argument of the main text, we can then define $\lambda_0^1 := \min\{\mathbb{G}_0(U_0), \mathbb{G}_1(U_0)\}$. Whatever the value of θ, agents will form private beliefs within this region with at least probability λ_0, where of course agents' private beliefs are conditionally independent. Hence the probability with which agent $\tilde{n}$ is

[26] That the agents here choose $x_n = 1$ is of course without loss of generality, we could have run through identical arguments assuming they all choose $x_n = 0$.

[27] Thanks to Sebastian Bervoets and Franz Ostrizek for posing this question to me.

a noise agent is in fact:

$$\underline{\rho}_1^1 := \frac{\rho}{\big(\rho + (1-\lambda_0^1)(1-\rho)\big)}$$

The major difference here is of course that upon observing a predecessor has chosen $x = 0$, agents immediately learn that this agent cannot be a noise agent. Hence, this probability is only relevant when interpreting the actions of neighbours who have chosen $x = 1$. Nonetheless, we can feed this back into the interval above to find U_0^2, generate a new λ_0^2, and a new ρ_2^1 just as in the main article. We can define $\underline{\rho}_\infty^1$ and λ_∞^1 through this process exactly as before, keeping in mind that $\underline{\rho}_\infty^1$ is now a lower bound on the probability with which any visible agent $\tilde{n}$ will be a noise agent, and relevant when forming beliefs about the type of a neighbour who chose $x = 1$, but irrelevant when observing $x = 0$.

Proposition 6 (Extending Unravelling Results)**.** *We can find equivalents of the following unravelling results:*

1. *Theorem 2: $\underline{\rho}_\infty^1$ is a lower bound on the probability that the $\tilde{n}^{th}$ visibly-acting agent is a noise type for any $\tilde{n} \in \mathbb{N}$, and the asymptotic fraction of agents that are noise types is almost surely greater than $\underline{\rho}_\infty^1$.*

2. *Proposition 5: Conditional on θ, accuracy can be bounded by*

$$\mathbb{P}(x_n = \theta | \theta = 1) \leq \left(1 - \mathbb{G}_1\left(1 - \frac{(1-\rho)^M}{(1-\rho)^M + \rho^M}\right)\right) \tag{E.1}$$

$$\mathbb{P}(x_n = \theta | \theta = 0) \leq (1 - \rho^M) + \rho^M \mathbb{G}_0(0.5) \tag{E.2}$$

$$\mathbb{P}(x_n = \theta) = 0.5\mathbb{P}(x_n = \theta | \theta = 0) + 0.5\mathbb{P}(x_n = \theta | \theta = 1) \tag{E.3}$$

3. *Corollary 2.1 simply updates the bound of Proposition 5 with $\underline{\rho}_\infty^1$. Clearly we can do similarly with our new bound.*

4. *Proposition 1 can be exactly restated, substituting $\underline{\rho}_\infty^1$ for $\underline{\rho}_\infty$ and λ_0^1 for λ_0.*

5. *Proposition 2: This result holds without adjustment.*

6. *Proposition 3: This result also holds without adjustment.*

Proof. To see the above points observe that:

1. Point 1 follows from the proof of Theorem 2 and the reasoning above, bearing in mind the changed definitions of ρ and λ.

2. Point 2: The portion of the upper bound on α_n with $\mathbb{G}_1$ can be derived exactly as in the proof of that statement, except that every $\rho/2$ can be replaced with a ρ, to bound accuracy conditional on $\theta = 1$. This gives Inequality E.1. A similar bound cannot be produced for accuracy conditional on $\theta = 0$, though one can observe that with probability ρ^M, they cannot perform better than $\mathbb{G}_0(0.5)$ (since with this probability their network is entirely noise, and they must underperform an agent

given no social information at all). This observation produces the second bound, Inequality E.2. One can substitute each of these (E.1 and E.2) into Equation E.3 for an overall bound on accuracy.

3. Point 4, the proof as written in Appendix C.3 stands, with only the two substitutions mentioned in the statement.

4. Point 5: As with the proof of Proposition 2, repeatedly shifting the t distribution to the right implies that λ_0^1 converges to 1 before even considering unravelling. This implies that $\underline{\rho}_{-\infty}^1 \to 1$, as can be seen in our above expression for $\underline{\rho}_{-1}^1 < \underline{\rho}_{-\infty}^1$. The rest of the proof of Proposition 2 can be applied without adjustment.

□

Therefore, this change to one-sided noise will still damage learning in sparse networks when $\theta = 0$, as can be seen in Points 2, 3 and 5 of Proposition 6, relative to the model with no noise agents. Without noise, as established in Theorem 3, we would achieve complete learning with unbounded beliefs. With these noise agents however, this is no longer the case, as any agent will, with some probability, still observe a neighbourhood of pure noise. Increasing the amount of timidity will also eventually completely smother learning (Point 5).

In dense networks, too, these noise agents can still help learning. The combined presence of noise agents and *enough* timidity, will ensure that the probability of observing a predecessor choosing $x = 1$ visibly will be different in the two states of the world. Here, however, the benefits to learning set out in Part 2 of Theorem 1 are lost. Since there are no noise agents who take action 0, it is now possible that there is some region of social beliefs in favour of $\theta = 1$ ($\lambda > 0.5$) such that no agent can receive a private signal strong enough to choose $x_n = 0$ *and* $v_n = 1$. With noise agents who uniformly randomise, the fact that any $\tilde{n}$ will always choose $x_{\tilde{n}} = 0$ with some probability ensures that the ratio of agents choosing $\{x_n = 1, v_n = 1\}$ against $\{x_n = 1, v_n = 0\}$ is observable; this means learning will continue. With one-sided noise however, this is no longer the case. We can resurrect learning here under a similar condition to that in the main text, only we now need one-sided unbounded beliefs (where they are unbounded in favour of $\theta = 0$, the state that is not being supported by bots).

Proposition 7 (Theorem 1 Part 2: Version with One-sided Noise)**.** *Suppose noise agents now always choose $x_n = 1$. If private beliefs have support $(0, \overline{b})$, i.e. they are bounded on the $\theta = 1$ side only, and the confidence distribution $\Delta(c)$ satisfies $[0, \epsilon) \cup (1 - \epsilon, 1) \subseteq \text{supp}(\Delta(c))$ for some $\epsilon > 0$, the social belief converges to certainty on the true state almost surely.*

Proof. Since private beliefs are unbounded in favour of $\theta = 0$, there are no cascade beliefs λ greater than 0.5. The upper bound on private beliefs $\overline{b}$ implies that social beliefs below $1 - \overline{b}$ are cascade beliefs in favour of $x_n = 0$. Agents acting at social beliefs below this point will choose $\{x_n = 1, v_n = 1\}$ with at least probability ρ (since all noise types choose $x_n = 1$). For any social belief below $1 - \overline{b}$, the probability with which agents at that social belief will choose $v_n = 1$ will be determined by the timidity distribution, and the distribution of private beliefs. Since the private beliefs are unbounded in favour of 0, $\mathbb{G}_0(\lambda) > \mathbb{G}_1(\lambda)$ for any $\lambda < 0.5$. Thus the probability with which the next visible action is 1 must be strictly lower if $\theta = 0$

than if $\theta = 1$ for any $\lambda < 0.5$. As we observed at the beginning of this proof, there are no cascade beliefs $\lambda \geq 0.5$. Hence this is true for all $\lambda \in [0, 1]$. As with the proof of Theorem 1, the result then follows from Smith and Sørensen (2000, Theorem B.1). □

That, in sparse networks, bots would actually do more damage to the cause they are promoting than to the other seems strange. A crucial feature of the model that drives this is the assumption that ρ and the behaviour of bots is common knowledge. If a foreign power sets up a series of bots to spout disinformation, in reality the effectiveness of such a move is clearly based on the fact that unsuspecting voters in the target country will omit to account for this fact, and simply be duped into believing that some of the bot-content they observe is legitimate. In my model, when the adversary chooses to programme enough bots to ensure 5% of all commenters are said bots, agents in the model are assumed automatically to know this. Introducing misspecification into the model would be necessary to capture the true impact of such a scheme, and would no doubt produce more failures of learning, as in Bohren and Hauser (2021). Nonetheless, even with common knowledge of ρ and bot-behaviour some damage is still done.

F Online Appendix: Convergence

Having studied the examples in this paper, the reader may be wondering under what circumstances we should expect the accuracy of visibly-acting agents to converge. The short answer is that we should not expect this to happen, but that the tractable immediate-successor networks with antisymmetric private beliefs I use in many examples will often produce convergence, though in Example 1 we can see it failing even here. In this appendix I set out the various factors that can prevent convergence.

In a line network with antisymmetric private beliefs, convergence is entirely a question of the gradient of the following function at and near the point it hits the 45-degree line:

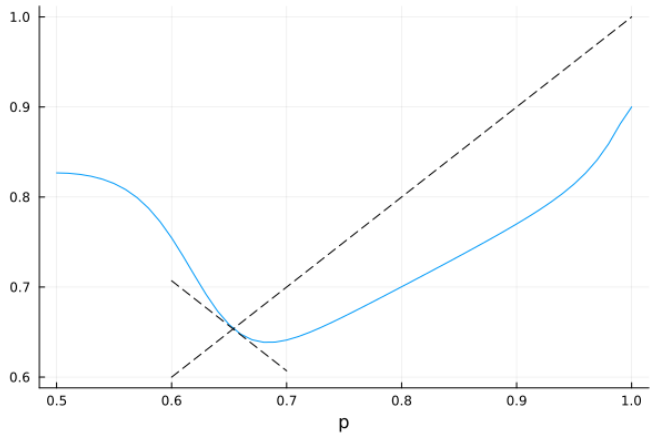


Figure 9: Accuracy as a function of neighbour accuracy, against a 45-degree line, with a -1 gradient tangent. The parameters are the same as in Figure 4a

This function plots the probability with which the visible agent (he) acting at a visible history at which he observes an action (of his predecessor, she) which matches the state with probability p then

matches the state. Unlike in Figure 7, this does not assume he is of social type (if it did, the blue curve would necessarily dominate the 45-degree line, as per Part 4 of Lemma 8). In a line network, whenever the blue line is above the 45-degree line, the visible agent in question outperforms his predecessor, and whenever it is below he underperforms her. Hence, we should expect the game to move towards the point at which the lines cross. Whether or not it actually converges, however, depends upon the gradient of the tangent at this crossing (this is a little reminiscent of the cobweb model of price fluctuations). If the blue line were simply a straight line with gradient -1 for a large enough region around the crossing, then from the moment the accuracy of an agent first fell within the domain of this segment it would begin infinitely fluctuating between two values forever. If the absolute value of the gradient of the blue line is greater, the accuracy of agents will explode away from the fixed point. If it is less, accuracy will converge. In this diagram, the gradient is steeper for x-values below the fixed point, and less steep above. Given this, whenever a neighbour is more accurate than at the fixed point, the observer will be less accurate but closer, and whenever she is less accurate he will be more accurate but further away. As graphed in Figure 4a, this converges to a recurrence between two points either side of the fixed point.

Hence even with otherwise perfect conditions for convergence, as shall be seen momentarily, it can still fail. More generally however, in any network topology where agents' neighbourhoods are of bounded size (bounded by some M), the simple fact of agents having neighbourhoods of different size will be enough to prevent convergence. For any $|B(n)| < M$, there is some probability that all of one's neighbours are noise agents, and the probability of the same being true for an agent with $|B(n)| + 1$ neighbours is of course smaller (assuming for contradiction that accuracy has converged to some α^*).

Even beyond this, in networks where agents do have the same number of neighbours, we can generate a contradiction when assuming that accuracy converges to α^* by supposing that the distance between one's neighbours follows one pattern if one's index is even, and another if it is odd. The distance between different neighbours will affect the correlation between their actions, and thus the value of the information.

Thus it can be seen that it is only in very regular networks such as k-immediate predecessor networks that one should expect convergence at all, and even then it is not guaranteed.

G Online Appendix: Line Example Working

Some of the graphs in this paper concern learning on immediate predecessor (or line) networks where the private signal structure induces antisymmetric private beliefs, and for these networks we can compute the probability with which an agent correctly guesses the state when their predecessor has accuracy α analytically. I include working for some examples of this here that I used to plot some of the figures above, so that interested readers can reproduce them as easily as possible.

G.1 Beta Private Signals and Pareto Timidity

- The signal densities are $\{2(1-s), 2s\}$.

- Timidity t follows a Pareto distribution[28] with $\alpha = 1$ and $t_{min} = 2$:

$$f_T(t) = \frac{2}{t^2}, \quad t \geq 2$$

This implies $(t-1)/t$ is distributed according to a uniform distribution on $[0.5, 1]$. Using $\tilde{z}_n$ as notation for the probability with which the n^{th} visible agent matches the state, we can write the following:

$$\tilde{z}_n := \mathbb{P}(\tilde{x}_{\tilde{n}} = \theta) = \mathbb{P}(\tilde{\tau}_n = N|\tilde{z}_{n-1})\big(\frac{1}{2}\big) + \mathbb{P}(\tilde{\tau}_{\tilde{n}} = S)(\tilde{z}_{n-1}) \times \mathbb{P}(\tilde{x}_{\tilde{n}} = \theta|\tilde{\tau}_{\tilde{n}} = S)(\tilde{z}_{n-1})$$

The social signal matches the state with probability $\tilde{z}_{n-1}$, suppose the true state is 1 and the social signal is 1, what are the relevant probabilities in this case?

The probability the next agent is a noise agent is:

$$\begin{aligned}
&\rho + (1-\rho) \times \mathbb{P}(p_n \in \{\text{Silent region for 1,1}\}) \times \rho \\
&+ \Big((1-\rho) \times \mathbb{P}(p_n \in \{\text{Silent...}\})\Big)^2 \times \rho + ... \\
&= \rho \sum_{j=0}^{\infty} \Big((1-\rho) \times \mathbb{P}(p_n \in \{\text{Silent ...}\})\Big)^j = \frac{\rho}{1 - \Big((1-\rho) \times \mathbb{P}(p_n \in \{\text{Silent...}\})\Big)}
\end{aligned}$$

Decision Rule Inequalities:

In $1,1$ (when $\theta = 1$ and $\tilde{x}_n = 1$), according to Lemma 2 an S agent gets the right answer visibly if $p_n\big(1 - \frac{(t-1)}{t}(1 - z_{n-1})\big) > 1 - z_{n-1}$. In $1,0$, an S agent gets the right answer visibly if the same left-hand expression is strictly greater than z_{n-1}. In $1,1$ they get the wrong answer visibly if $p_n(1 + (t-1)z_{n-1}) < (1 - z_{n-1})$ and in $1,0$ they get it wrong visibly if this left-hand expression is strictly less than z_{n-1}.

Since the private beliefs are antisymmetric, we can just consider the case in which $\theta = 1$. As can be understood from Lemma 4, the relevant probabilities are symmetric. Next, we must compute the distributions of the expressions on the left hand side of each of the above inequalities, of which two of these expressions are distinct. In each case, Jacobian transformation and convolution formulae suffice to compute the probability density functions.

One can compute that the probability density function of a variable distributed as $p_n \times (1 - \frac{t-1}{t} z_{n-1})$ is:

$$f_{VR}(\zeta) = \begin{cases} \frac{4\zeta}{c}\left(\frac{1}{1-c} - \frac{1}{1-0.5c}\right), & 0 \leq \zeta \leq 1-c, \\ \frac{4}{c}\left(1 - \frac{\zeta}{1-0.5c}\right), & 1-c \leq \zeta \leq 1-0.5c, \\ 0, & \text{otherwise.} \end{cases}$$

[28] A Pareto distribution has pdf:

$$f_{Pareto}(x) = \frac{\alpha x_{\min}^{\alpha}}{x^{\alpha+1}}, \quad x \geq x_{\min},$$

where of course zero density is assigned below the min parameter.

Similarly one distributed as $p_n\Big(1+(t-1)(1-z_{n-1})\Big)$ has density function:

$$f_{VW}(\zeta)=\begin{cases}0, & \zeta\leq 0,\\ \dfrac{4c\zeta}{(1-c)^3}\left[\dfrac{1+c}{2c}-\dfrac{2c}{1+c}-2\ln\left(\dfrac{1+c}{2c}\right)\right], & 0<\zeta<1+c,\\ \dfrac{4c}{(1-c)^3}\left\{\dfrac{\zeta^2}{\zeta-(1-c)}+2\zeta\ln\left(\dfrac{\zeta-(1-c)}{\zeta}\right)-\Big[\zeta-(1-c)\Big]\right\}, & \zeta\geq 1+c.\end{cases}$$

To compute the probabilities that the above probabilities are satisfied we need the CDFs, which are unwieldy to write out and therefore omitted, but serve to calculate the probabilities that our above inequalities hold:

- In $1,1$, c takes value z_{n-1} in the above, and we want probabilities that $\zeta\geq 1-z_{n-1}$.
- In $1,0$, c takes value $(1-z_{n-1})$ in both of these, and we want probabilities that $\zeta\geq z_{n-1}$.

$$F_{VR11}(1-z_{n-1})=\frac{(1-z_{n-1})^2}{(1-z_{n-1})(1-0.5z_{n-1})}=\frac{(1-z_{n-1})}{(1-0.5z_{n-1})}$$

$$\begin{aligned}F_{VW11}(1-z_{n-1})&=\frac{(1-z_{n-1})^2}{2}\Big(\frac{4z_{n-1}}{(1-z_{n-1})^3}\left[\frac{1+z_{n-1}}{2z_{n-1}}-\frac{2z_{n-1}}{1+z_{n-1}}-2\ln\left(\frac{1+z_{n-1}}{2z_{n-1}}\right)\right]\Big)\\&=\frac{2z_{n-1}}{(1-z_{n-1})}\left[\frac{1+z_{n-1}}{2z_{n-1}}-\frac{2z_{n-1}}{1+z_{n-1}}-2\ln\left(\frac{1+z_{n-1}}{2z_{n-1}}\right)\right]\end{aligned}$$

$$F_{VR10}(z_{n-1})=\frac{z_{n-1}^2}{(1-1+z_{n-1})(1-0.5(1-z_{n-1}))}=\frac{2z_{n-1}}{1+z_{n-1}}$$

$$\begin{aligned}F_{VW10}(z_{n-1})&=\frac{z_{n-1}^2}{2}\frac{4(1-z_{n-1})}{(z_{n-1})^3}\left[\frac{2-z_{n-1}}{2(1-z_{n-1})}-\frac{2(1-z_{n-1})}{2-z_{n-1}}-2\ln\left(\frac{2-z_{n-1}}{2(1-z_{n-1})}\right)\right]\\&=\frac{2(1-z_{n-1})}{(z_{n-1})}\left[\frac{2-z_{n-1}}{2(1-z_{n-1})}-\frac{2(1-z_{n-1})}{2-z_{n-1}}-2\ln\left(\frac{2-z_{n-1}}{2(1-z_{n-1})}\right)\right]\end{aligned}$$

And we can finally begin plugging-in these probabilities to compute the conditional probabilities (conditional on the social signal), which will then allow us to compute the unconditional probabilities required in our original equation:

Probability of being Noise agent in $1,1$:

$$\begin{aligned}&\mathbb{P}(\tilde{\tau}_n=N|\theta=1,\tilde{x}_{n-1}=1,z_{n-1})=\\&=\frac{\rho}{1-(1-\rho)\big(1-(1-F_{VR11}(1-z_{n-1})+F_{VW11}(1-z_{n-1}))\big)}\\&=\frac{\rho}{1-(1-\rho)\big(F_{VR11}(1-z_{n-1})-F_{VW11}(1-z_{n-1})\big)}\\&=\frac{\rho}{1-(1-\rho)\Big(\frac{1-z_{n-1}}{1-0.5z_{n-1}}-\frac{2z_{n-1}}{1-z_{n-1}}\Big[\frac{1+z_{n-1}}{2z_{n-1}}-\frac{2z_{n-1}}{1+z_{n-1}}-2\ln\Big(\frac{1+z_{n-1}}{2z_{n-1}}\Big)\Big]\Big)}\end{aligned}$$

Probability of being Noise agent in $1,0$:

$$\begin{aligned}
&\mathbb{P}(\tilde{\tau}_n = N|\theta = 1, \tilde{x}_{n-1} = 0, z_{n-1}) = \\
&= \frac{\rho}{1-(1-\rho)\big(F_{VR10}(z_{n-1}) - F_{VW10}(z_{n-1})\big)} \\
&= \frac{\rho}{1-(1-\rho)\Big(\frac{2z_{n-1}}{1+z_{n-1}} - \frac{2(1-z_{n-1})}{z_{n-1}}\Big[\frac{2-z_{n-1}}{2(1-z_{n-1})} - \frac{2(1-z_{n-1})}{2-z_{n-1}} - 2\ln\Big(\frac{2-z_{n-1}}{2(1-z_{n-1})}\Big)\Big]\Big)}
\end{aligned}$$

Probability of $\tilde{n}$ being visibly correct conditional on $\tilde{\tau}_n = S$ agent in $1,1$:

$$\begin{aligned}
&\mathbb{P}(\tilde{x}_n = \theta \mid \tilde{\tau}_n = S, \theta = 1, \tilde{x}_{n-1} = 1) = \frac{1 - F_{VR11}(1-z_{n-1})}{1 - F_{VR11}(1-z_{n-1}) + F_{VW11}(1-z_{n-1})} \\
&= \frac{1 - \frac{1-z_{n-1}}{1-0.5z_{n-1}}}{1 - \frac{1-z_{n-1}}{1-0.5z_{n-1}} + \frac{2z_{n-1}}{1-z_{n-1}}\Big[\frac{1+z_{n-1}}{2z_{n-1}} - \frac{2z_{n-1}}{1+z_{n-1}} - 2\ln\Big(\frac{1+z_{n-1}}{2z_{n-1}}\Big)\Big]}
\end{aligned}$$

Probability of $\tilde{n}$ being visibly correct conditional on $\tilde{\tau}_n = S$ agent in $1,0$:

$$\begin{aligned}
&\mathbb{P}(\tilde{x}_n = \theta \mid \tilde{\tau}_n = S, \theta = 1, \tilde{x}_{n-1} = 0) = \frac{1 - F_{VR10}(z_{n-1})}{1 - F_{VR10}(z_{n-1}) + F_{VW10}(z_{n-1})} \\
&= \frac{1 - \frac{2z_{n-1}}{1+z_{n-1}}}{1 - \frac{2z_{n-1}}{1+z_{n-1}} + \frac{2(1-z_{n-1})}{z_{n-1}}\Big[\frac{2-z_{n-1}}{2(1-z_{n-1})} - \frac{2(1-z_{n-1})}{2-z_{n-1}} - 2\ln\Big(\frac{2-z_{n-1}}{2(1-z_{n-1})}\Big)\Big]}
\end{aligned}$$

The Unconditional Quantities of Interest:

$$\begin{aligned}
\mathbb{P}(\tilde{\tau}_n = N|z_{n-1}) &= \mathbb{P}(\tilde{\tau}_n = N|\theta = 1, z_{n-1}) && \text{By symmetry} \\
&= z_{n-1}\mathbb{P}(\tilde{\tau}_n = N|\theta = 1, \tilde{x}_{n-1} = 1, z_{n-1}) \\
&\quad + (1 - z_{n-1})\mathbb{P}(\tilde{\tau}_n = N|\theta = 1, \tilde{x}_{n-1} = 0, z_{n-1})
\end{aligned}$$

The unconditional probability of success assuming an agent is social is:

$$\begin{aligned}
\mathbb{P}(\tilde{x}_n = \theta|\tilde{\tau}_n = S) = z_{n-1}&\mathbb{P}(\tilde{x}_n = \theta \mid \tilde{\tau}_n = S, \theta = 1, \tilde{x}_{n-1} = 1) \\
&+ (1 - z_{n-1})\mathbb{P}(\tilde{x}_n = \theta \mid \tilde{\tau}_n = S, \theta = 1, \tilde{x}_{n-1} = 0)
\end{aligned}$$

Substituting these into our earlier expressions in here we get closed form expressions for $\mathbb{P}(\tilde{\tau}_n = N|z_{n-1})$ and $\mathbb{P}(\tilde{x}_n = \theta|\tilde{\tau}_n = S)$, which can in turn be substituted into our original expression:

$$\tilde{z}_n := \mathbb{P}(\tilde{x}_n = \theta) = \mathbb{P}(\tilde{\tau}_n = N|\tilde{z}_{n-1})\big(\frac{1}{2}\big) + \mathbb{P}(\tilde{\tau}_n = S|\tilde{z}_{n-1}) \times \mathbb{P}(\tilde{x}_n = \theta|\tilde{\tau}_n = S, \tilde{z}_{n-1})$$

Since these are very messy expressions, I shall omit to actually write them out here. Nonetheless, plotting the relevant curves illustrates their properties. When I include figures from this example in the body of the main article, I am simply plotting these curves.

G.2 Poisson Timidity and Beta Signals

In this subsection I consider when $t-1$ is distributed according to a Poisson distribution with parameter λ. In this instance, one can compute that the two relevant pdfs are:

$$f_{VR}(\zeta) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \frac{2\zeta}{\left(1 - \frac{ck}{k+1}\right)^2} \cdot \mathbf{1}\left(0 \leq \zeta \leq 1 - \frac{ck}{k+1}\right)$$

$$f_{VW}(\zeta) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \frac{2\zeta}{(1+ck)^2} \cdot \mathbf{1}\left(0 \leq \zeta \leq 1 + ck\right)$$

Following the same steps as before this produces the four CDFs we need:

$$F_{VR11}(\zeta) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \begin{cases} \left(\frac{\zeta}{1 - \frac{z_{n-1}k}{k+1}}\right)^2 & \text{if } \zeta < 1 - \frac{z_{n-1}k}{k+1} \\ 1 & \text{if } \zeta \geq 1 - \frac{z_{n-1}k}{k+1} \end{cases}$$

$$F_{VR10}(\zeta) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \begin{cases} \left(\frac{\zeta}{1 - \frac{(1-z_{n-1})k}{k+1}}\right)^2 & \text{if } \zeta < 1 - \frac{(1-z_{n-1})k}{k+1} \\ 1 & \text{if } \zeta \geq 1 - \frac{(1-z_{n-1})k}{k+1} \end{cases}$$

$$F_{VW11}(\zeta) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \begin{cases} \left(\frac{\zeta}{1 + z_{n-1}k}\right)^2 & \text{if } \zeta < 1 + z_{n-1}k \\ 1 & \text{if } \zeta \geq 1 + z_{n-1}k \end{cases}$$

$$F_{VW10}(\zeta) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \begin{cases} \left(\frac{\zeta}{1 + (1-z_{n-1})k}\right)^2 & \text{if } \zeta < 1 + (1 - z_{n-1})k \\ 1 & \text{if } \zeta \geq 1 + (1 - z_{n-1})k \end{cases}$$

Substituting these values into the expressions we have already derived allows us to plot the graphs with the parameters distributed according to these new distributions. Figures 5a, 5b and 8a are computed using these quantities.

G.3 Normal Signals

Here I record formulae that give the distribution of private beliefs when the private signals are distributed normally. I do not go through the same analytic computation as above as it is too laborious. Instead I have simply written some code to compute the thresholds within which a private signal produces an invisible action for each possible neighbour-action, and work out the required probabilities directly.

G.3.1 Antisymmetric Case ($\sigma_0 = \sigma_1$)

Let μ_0, μ_1 be the signal means in state 0 and 1 respectively, assuming the variance of each is 1. Then, the probability that an agent forms a posterior belief less than or equal to p, given that the true state is

0, is:

$$\mathbb{P}_{\theta=0}\left(\mathbb{P}(\theta=1 \mid s) \leq p\right) = \Phi\left(\frac{1}{2(\mu_0-\mu_1)}\left[2\log\left(\frac{1-p}{p}\right) - (\mu_1^2-\mu_0^2)\right] - \mu_0\right)$$

where Φ is the standard normal CDF. If, to reduce the number of parameters without the loss of any degrees of freedom, we can set the signal means as $\mu_0 = -\mu$, $\mu_1 = \mu$, we have:

$$\mathbb{P}_{\theta=1}\left(\mathbb{P}(\theta=1 \mid s) \leq p\right) = \Phi\left(-\frac{\log\left(\frac{1-p}{p}\right)}{2\mu} - \mu\right)$$

$$\mathbb{P}_{\theta=0}\left(\mathbb{P}(\theta=1 \mid s) \leq p\right) = \Phi\left(-\frac{\log\left(\frac{1-p}{p}\right)}{2\mu} + \mu\right)$$

G.3.2 Non-antisymmetric case

Now suppose the variances of the two signals are not the same (which allows us to generate non-antisymmetric private beliefs). They are now distributed as:

$$s \mid \theta = i \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

where I further assume that $\mu_0 = -\mu$, $\mu_1 = \mu$ for some $\mu > 0$.

We wish to compute:

$$\mathbb{P}_{\theta=1}\left(\mathbb{P}(\theta=1 \mid s) \leq p\right)$$

This is equivalent to solving:

$$\log\left(\frac{f_1(s)}{f_0(s)}\right) \leq \log\left(\frac{p}{1-p}\right)$$

where the f functions are the density functions of the signals.

The log-likelihood ratio is:

$$\log\left(\frac{f_1(s)}{f_0(s)}\right) = \log\left(\frac{\sigma_0}{\sigma_1}\right) - \frac{(s-\mu)^2}{2\sigma_1^2} + \frac{(s+\mu)^2}{2\sigma_0^2}$$

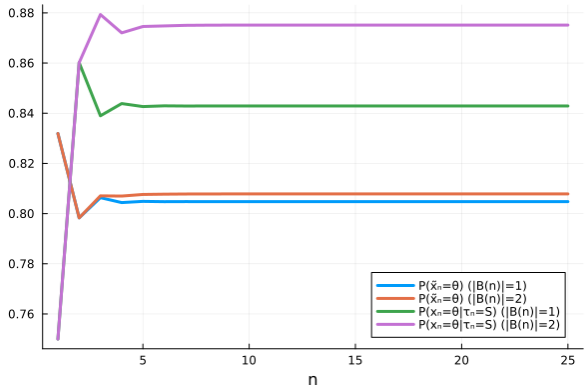
Hence the probability of a private belief lower than p is the probability that the following quadratic inequality holds:

$$as^2 + bs + c \leq 0$$

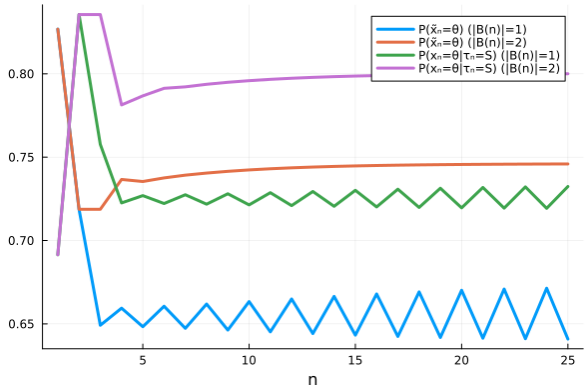
where

$$\begin{aligned} a &= \frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2} \\ b &= \frac{\mu}{\sigma_0^2} + \frac{\mu}{\sigma_1^2} \\ c &= \log\left(\frac{\sigma_0}{\sigma_1}\right) + \frac{\mu^2}{2\sigma_0^2} - \frac{\mu^2}{2\sigma_1^2} - \log\left(\frac{p}{1-p}\right) \end{aligned}$$

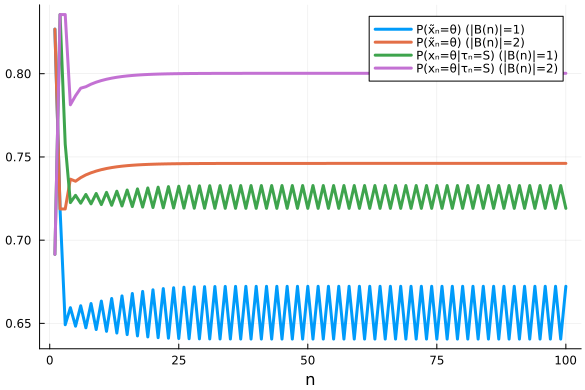
Let s_1, s_2 be the roots of the quadratic. Then:



(a) With signal structure $\{2(1-s), 2s\}$.



(b) With signal structure $\{N(0,1), N(1,1)\}$



(c) Normal but showing up to $n/\tilde{n} = 100$

Figure 10: The parameters for these graphs are the same as in Example 1, except that I plot both the immediate predecessor network and the 2-immediate predecessor network.

$$\mathbb{P}_{\theta=1}\left(\mathbb{P}(\theta = 1 \mid s) \leq p\right) = \Phi\left(\frac{\max(s_1, s_2) - \mu}{\sigma_1}\right) - \Phi\left(\frac{\min(s_1, s_2) - \mu}{\sigma_1}\right)$$

As with the normal-signal antisymmetric case, we can then compute the required thresholds using this. Figure 8b is computed with this procedure (using the parameters specified just above it). Since this computational method does not require the same simplifying assumptions as the analytical case, I also use it to study what happens in a 2-immediate predecessor network. Doing this has produced no important insights so I exclude them from the main article, but here I present some graphs showing how the Example 1 graphs in Figure 4a change. The first two agents are always identical in these graphs since the network topology is identical for them, but in the 2-immediate predecessor agents perform better asymptotically.